THE UNIVERSITY OF QUEENSLAND

AUSTRALIA

RICCI FLOW ON HOMOGENEOUS SUPERMANIFOLDS

BENJAMIN KRUGER

SUPERVISOR: ARTEM PULEMOTOV
CO-SUPERVISOR: YANG ZHANG

SUBMITTED FOR THE DEGREE OF
BACHELOR OF ADVANCED SCIENCE (HONOURS)
NOVEMBER, 2023

THE UNIVERSITY OF QUEENSLAND
SCHOOL OF MATHEMATICS AND PHYSICS

Acknowledgements

First and foremost, I extend my gratitude to my supervisor, Artem Pulemotov, for his unwavering support and invaluable guidance throughout not only this project but also my undergraduate degree as a whole.

I would also like to thank Yang Zhang, my associate supervisor, whose patience and expertise shone in our countless discussions. Yang's dedication to helping me understand various topics has significantly enriched both the quality of this document and my growth as a mathematician.

I would like to thank Ramiro Lafuente and Timothy Buttsworth for their insightful discussions and suggestions on the more technical aspects of the thesis. I am grateful to my friends at the University of Queensland, who made finding time to work on my thesis difficult amongst all the coffee breaks. A special thanks goes to Joshua Peters and Adam Thompson for their helpful discussions working through various arguments.

To my family and friends outside of mathematics, I extend my thanks for their genuine interest in my work. Lastly, I thank my partner, Anneliese, for enduring (what were undoubtedly one-sided and occasionally dull) conversations about my thesis and for helping make time away from mathematics more enjoyable.

Contents

CHAPTER 1

Introduction

Symmetry plays a fundamental role in physics. One perspective views symmetry as the property of remaining unchanged under a transformation. A familiar example of this is the rotational symmetry displayed by a sphere. This is an example of a continuous symmetry, as opposed to a discrete symmetry. The appreciation of continuous symmetries grew after a result of Noether [Noe18], which roughly says that continuous symmetries correspond to conserved quantities. For example, rotational symmetry corresponds to the conservation of angular momentum [Gri13]. Mathematically, continuous symmetries are described through the action of Lie groups on objects. In the case of the sphere, its rotational symmetry is captured by the special orthogonal group $\mathsf{SO}(3)$.

Modern physics mainly deals with two types of symmetry: spacetime symmetries and internal symmetries. Until 1967, there was an effort to combine the two into a single unifying symmetry. However, this task was deemed impossible following the work of Coleman and Mandula [Col67]. In essence, they showed that there is no non-trivial way to combine the two symmetries. However, this only applies to bosonic symmetries (those linked to force-carrying particles). An extension introduced by Haag, Łopuszański, and Sohnius [HŁS75] led to the incorporation of a new fermionic symmetry, establishing a relationship between bosons (force-carrying particles) and fermions (matter-carrying particles).

The introduction of this fermionic symmetry lead to the theory of *supersymmetry*. A prediction of supersymmetry is the existence of 'partner' particles: corresponding to each boson (resp. fermion), there would be a partner fermionic (resp. bosonic) particle. An example is the electron and its partner, the selectron [Rog07]. This prediction offers reasoning for the mass of the Higgs boson. Beyond this, supersymmetry has relevance in string theory and addresses the 'hierarchy problem' in the standard model.

The theory of supersymmetry relies on supergeometric objects. One of the most fundamental of these objects is the supermanifold. Supermanifolds have been defined independently by many authors [DeW84, Kos77, Lei80, Rog80]. A supermanifold generalises the notion of a manifold by introducing a coordinate system with both commuting and anti-commuting coordinates. Intuitively, these extra coordinates allow descriptions of both bosons and fermions in the same space.

There are two main constructions of supermanifolds: the concrete construction, and the algebro-geometric construction. The concrete construction describes supermanifolds as spaces that, locally, resemble flat superspace (the model space in supergeometry). On the other hand, the algebro-geometric construction defines a supermanifold as a topological space equipped with a sheaf of $\mathbb{Z}/2\mathbb{Z}$-graded algebras. This is analogous to how we can view a smooth manifold as a ringed space that is locally isomorphic to Euclidean space with its sheaf of smooth functions.

One outcome of developing the theory of supersymmetry is a generalisation of the current mathematical framework in physics. The existing local model of the universe, in general relativity, can be described as a pseudo-Riemannian manifold. So, it is natural to consider Riemannian structures on supermanifolds. The field of Riemannian geometry enables one to consider angles, length and volume on geometric objects. A key concept in the field is that of a Riemannian metric. A Riemannian metric endows a smooth manifold with a family of inner products on its tangent spaces. It has been shown that a generalisation of this, and many other concepts in Riemannian geometry, can be defined for supermanifolds; for instance, see [Goe08].

This project focuses on investigating the Ricci flow in the setting of supermanifolds. In particular, we consider homogeneous superspaces.

The Ricci flow is a second-order partial differential equation that evolves a Riemannian metric over time. Intuitively, the Ricci flow deforms a Riemannian metric towards one with distinguished curvature. For example, if $n = 2$, the Ricci flow will deform a metric to one with constant curvature, giving a proof of the two-dimensional uniformisation theorem.

Let (M, g_0) be a Riemannian manifold of dimension n. A one-parameter family of Riemannian metrics $\{g(t)\}_{t\in[0,T)}$ is said to be a *Ricci flow* with initial metric g_0 if it satisfies the initial value problem

$$\frac{\partial}{\partial t}g(t) = -2\,\mathrm{Ric}(g(t)), \quad g(0) = g_0. \tag{1.1}$$

The Ricci flow was first introduced by Hamilton in [Ham82], where he proved that a compact three-manifold which admits a metric with strictly positive Ricci curvature also admits a metric of constant positive curvature. The most notable application of Hamilton's Ricci flow is in Perelman's proof of the Poincaré conjecture (and the more general Thurston's conjecture) in 2002 and 2003. The Poincaré conjecture asserts that every simply connected, closed (compact and without boundary) three-manifold is homeomorphic to the three-sphere.

To gain some insight into the motivation behind equation (1.1), fix a point $p \in M$. At p, one can choose coordinates so that

$$R_{ij} = \Delta(g_{ij}) + \text{ lower order terms},$$

where Δ is the Laplace-Beltrami operator, a generalisation of the Laplacian to functions on Riemannian manifolds. In this light, (1.1) looks remarkably similar to the heat equation

$$\frac{\partial}{\partial t}u(t) = \Delta u.$$

The heat equation can be understood as a process that disperses heat over time. This analogy motivates the intuition described above, where the Ricci flow 'averages out' the metric according to its Ricci curvature. However, the Ricci flow is not precisely a heat flow, as it only demonstrates weakly parabolic behaviour. As a result, the conventional theory of existence and uniqueness of parabolic equations cannot be immediately applied. Regardless, short-time existence and uniqueness was established by Hamilton for compact manifolds using the Nash-Moser inverse function theorem [Ham82]. DeTurk subsequently presented a more straightforward proof where he considered a strictly parabolic equation 'equivalent' to (1.1). This facilitated the application of the standard theory of existence and uniqueness of parabolic equations [DeT83].

A driving philosophical question in Riemannian geometry is whether, for a fixed manifold M, there exists a 'best' metric on M [Bes87, 0.4]. In dimension two, this question is answered by the metrics of constant Gauss curvature. A natural generalisation of this in higher dimensions is the notion of constant Ricci curvature [Bes87, 0.6]. Such metrics are called Einstein and are characterised by the equation $\mathrm{Ric}\, g = \lambda g$, $\lambda \in \mathbb{R}$. Among other distinguished properties, Einstein metrics evolve only by scaling under the Ricci flow. Indeed, suppose the initial metric g_0 is Einstein. A solution to (1.1) gives

$$g(t) = (1 - 2\lambda t)g_0.$$

We see different behaviour in the flow based on the sign of λ. For example, the round sphere $(\mathbb{S}^n, g_0)$ has $\mathrm{Ric}(g_0) = (n-1)g_0$, so $g(t) = (1-2(n-1)t)g_0$. As $t \to T = \frac{1}{2(n-1)}$, the space collapses to a point but retains its 'shape'. This is known as the round shrinking sphere.

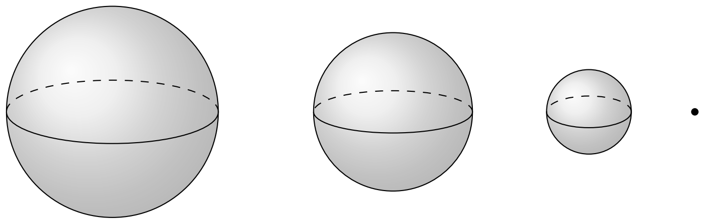

FIGURE 1. The evolution of $\mathbb{S}^2$ under the Ricci flow

On the other hand, hyperbolic space $(\mathbb{H}^n, g_0)$ has $\mathrm{Ric}(g_0) = -(n-1)g_0$, so $g(t) = (1+2(n-1)t)g_0$. In this case, we say the space expands homothetically for all time. If $\lambda = 0$ then the initial metric is fixed along the flow.

In general, the topology of the manifold does not always allow for the existence of such distinguished metrics. In these cases, the Ricci flow may develop a singularity. A singularity is reached at time T if the flow cannot be smoothly extended beyond T. We have already encountered an example of this in the round shrinking sphere. The good news is that singularities can be overcome. One technique for this involves renormalising the Ricci flow. For instance, to keep the volume of $(M, g(t))$ constant over time, we consider the normalised Ricci flow equation

$$\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Ric}\, g(t) + \frac{2}{n} \frac{\int_M S(g(t))\mathrm{dvol}_g}{\int_M \mathrm{dvol}_g} g(t).$$

Under this equation, the round sphere, for example, is a genuine fixed point.

We now specialise the discussion to G-invariant metrics on homogeneous spaces. In this setting, the Ricci flow equation simplifies to a system of ordinary differential equations (ODEs).

Consider a group G acting on a smooth manifold M. A Riemannian manifold (M, g) is *homogeneous* if G is a closed subgroup of the isometry group of M that acts transitively. Loosely, this condition says that M has the same geometry at every point. It can be shown that a Riemannian homogeneous space (M, g) is diffeomorphic to the quotient space G/H. Here, H represents the isotropy subgroup G_p at some $p \in M$ which, since G acts transitively, is conjugate to $G_{p'}$ for all $p' \in M$.

If G/H admits a G-invariant metric, then the Lie algebra of G has the decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \tag{1.2}$$

where $\mathfrak{h} = \mathrm{Lie}(H)$ and $\mathfrak{m}$ is the module associated with the isotropy representation. We have the remarkable one-to-one correspondence:

$$\left\{ \begin{array}{c} \text{G invariant metrics,} \\ g, \text{ on } G/H \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \mathrm{Ad}^H\text{-invariant scalar products} \\ \langle \cdot, \cdot \rangle \text{ on } \mathfrak{m} \end{array} \right\}.$$

This correspondence changes the problem of studying G-invariant metrics on G/H to studying Ad^H-invariant scalar products on the module $\mathfrak{m}$.

The Ricci flow on homogeneous spaces, as well as the related problem concerning the existence of Einstein metrics on the same spaces, has been studied extensively [BWZ04, Böh15, Buz14, DK08, IJ92, Laf15, WZ86, WZ91]. Wang and Ziller [WZ86] apply a variational approach to present a general existence theorem for homogeneous Einstein metrics, while also providing examples of homogeneous spaces with no Einstein metrics. Restricting to G-invariant metrics with volume one, they consider the critical points of the Einstein-Hilbert functional

$$\mathscr{E}(g) = \int_M S(g)\mathrm{dvol}_g,$$

and show that these are precisely the Einstein metrics on M. Given a homogeneous space G/H, it is not always guaranteed that a homogeneous Einstein metric exists. The lowest-dimensional example of this, with dimension 12, is $\mathsf{SU}(4)/\mathsf{SU}(2)$.

In the case where $\mathfrak{m}$ decomposes into two irreducible, inequivalent summands $\mathfrak{m}_1$ and $\mathfrak{m}_2$, Buzano completely describes the behaviour of the homogeneous Ricci flow [Buz14]. It was shown that the homogeneous Ricci flow reaches a singularity in finite time. Böhm studies the long-time behaviour of homogeneous Ricci flows in general [Böh15]. He shows that on any homogeneous space not diffeomorphic to the torus T^n, the Ricci flow reaches a singularity in finite time.

We now turn our attention to the super setting. Let $\mathscr{M} = G/H$ be a homogeneous superspace. Following the non-super analogue, we establish a decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}.$$

In this project, we investigate scenarios where $\mathfrak{m}$ breaks down into s irreducible summands

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_s.$$

We choose this decomposition in such a way that our G-invariant metric g decomposes as

$$\langle \cdot, \cdot \rangle = x_1 Q|_{\mathfrak{m}_1} \oplus \cdots \oplus x_s Q|_{\mathfrak{m}_s}, \tag{1.3}$$

where $x_i \in \mathbb{R} \setminus \{0\}$ for all $1 \leq i \leq s$ and Q is a fixed Riemannian metric on G. It turns out that a solution $g(t)$ to the Ricci flow (1.1) starting at g remains homogeneous and has decomposition

$$\langle \cdot, \cdot \rangle = x_1(t) Q|_{\mathfrak{m}_1} \oplus \cdots \oplus x_s(t) Q|_{\mathfrak{m}_s},$$

where each $x_i(t)$ is a smooth function of t with $x_i(0) = x_i$.

Our focus first lies in the case when $s = 1$, i.e. $\mathfrak{m}$ is irreducible. As observed in the non-super context, Schur's lemma asserts that there exists a unique, up to scaling, G-invariant metric on G/H, which is necessarily Einstein. A key contrast to the non-super setting is that certain quantities (such as structure constants or the dimension of the summands) can take on negative values. As a result, we see one of two behaviours in the Ricci flow:

(i) $x(t) \to 0$ as $t \to T$, or
(ii) $x(t) \to \pm\infty$ as $t \to \infty$.

We subsequently consider when $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$. In particular, we assume that H is not maximal in G. In this setting, we make some progress, leading to a conjecture:

CONJECTURE A. *Let G/H be a homogeneous superspace where H is not maximal in G and consider a homogeneous G-invariant Riemannian metric of the form (1.3). If $(x_1(0), x_2(0)) \in \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \neq 0, x_2 > 0\}$, then the behaviour of the quantity $\frac{x_1(t)}{x_2(t)}$ under the homogeneous super Ricci flow is as in Tables 2, 3, and 4.*

On the other hand, if $x_2(0) < 0$ and $A+B > 0$ (resp. $A+B < 0$), then the behaviour of $\frac{x_1(t)}{x_2(t)}$ under the homogeneous super Ricci flow is as in Tables 2, 3, and 4, found on pages 49 and 50, corresponding to the columns with $A+B < 0$ (resp. $A+B > 0$).

This conjecture, if proven true, says that the Ricci flow on homogeneous supermanifolds exhibits different behaviour, in general, to the non-super counterpart. In support of this, we construct two infinite families of homogeneous supermanifolds with two inequivalent irreducible isotropy summands and study the Ricci flow of their G-invariant metrics.

In our first family of examples, $G/H = \mathsf{SU}(pq+m|n)/\mathsf{SU}(p) \times \mathsf{SU}(q) \times \mathsf{U}(m|n)$, the Ricci flow has no finite time singularities given $x_1(0), x_2(0) > 0$ and $m > n$.

We study the Ricci flow of G-invariant metrics on the homogeneous superspace $G/H = \mathsf{SOSp}(2|2n)/\mathsf{SOSp}(2|2(p-1)) \times \mathsf{Sp}(2(n+1-p))$ in full generality. We consider multiple cases based on the initial signature of the metric.

Structure of the thesis

This thesis is divided into four main chapters.

In chapter 2, we introduce a grading to some basic objects, a key feature of supergeometry. First, we investigate the grading of vector spaces, algebras and modules. We explore the language of sheaves from algebraic geometry, which we then use to define supermanifolds and the structures on them. This is followed by extending some basic notions from differential geometry, such as vector fields and maps between manifolds, to the super setting. We introduce the theory of Lie supergroups and their actions on supermanifolds, ending the chapter with a brief review of the representation theory of Lie groups. The exposition in this chapter draws from [CCF11, EH00, Har77, Kos77, Lei80, Var04].

Chapter 3 introduces the additional structure of a graded Riemannian metric to supermanifolds. Applying this, we define connections and curvature, seeing an analogue of the fundamental theorem of Riemannian geometry hold. We finish by defining homogeneous superspaces and discussing their geometry. References for this chapter include [DM99, DeW84, Goe08, Kac77, Sch84].

In Chapter 4, we discuss some of the more technical background of the Ricci flow, developing tools that will be useful for our analysis of homogeneous spaces. We then set the scene for the Ricci flow on homogeneous spaces, outlining the known results and theory in the field. The end of this chapter presents a detailed example of the Ricci flow on a low-dimensional Lie supergroup to highlight some key differences in computation. This chapter follows information presented in [Arv03, Bes87, BWZ04, Böh15, Buz14, Ham82, Ham84, IJ92, Laf15, Pet06, WZ86, WZ91].

In Chapter 5, we introduce the homogeneous Ricci flow on supermanifolds, establishing notation and conventions. We then discuss an obstruction to the variational interpretation in the super setting. The remainder of the chapter focuses on the analysis of singularities in the homogeneous Ricci flow, specifically for spaces whose isotropy representation decomposes into one or two irreducible summands. We conclude by discussing future directions.

CHAPTER 2

An introduction to supergeometry

In this chapter, we introduce the basic definitions and essential results necessary for the study of supergeometry. A succinct overview of Lie group representation theory is also included at the end of this chapter. For readers seeking a comprehensive introduction to these topics, we suggest consulting [Arv03, CCF11, Kos77, Lei80, Var04].

2.1. Super linear algebra

This section introduces the concept of a $\mathbb{Z}/2\mathbb{Z}$-grading in linear algebra, with adaptations of some familiar definitions in the field.

One of the first definitions encountered is that of a vector space. A $\mathbb{Z}/2\mathbb{Z}$-grading (typically labelled with the prefix 'super') distinguishes objects based on their parity. In this context, a *vector superspace* is a vector space that splits into two subspaces

$$V = V_0 \oplus V_1,$$

where $v \in V_0$ is considered even and $v \in V_1$ is considered odd. Elements that lie exclusively in V_0 or V_1 are called *homogeneous*. The *parity* of such elements is defined as:

$$|v| = \begin{cases} 0, & v \in V_0 \\ 1, & v \in V_1. \end{cases}$$

If the subspaces V_0 and V_1 have dimensions m and n, respectively, we say V has dimension $m|n$. To specify the even and odd dimensions, we sometime write $V^{m,n}$.

EXAMPLE 2.1. *The set of all linear maps between two supervector spaces, V and W, forms a vector superspace, which we denote by* $\mathrm{Hom}(V,W)$. *This space can be divided into maps that preserve parity and reverse it:*

$$\mathrm{Hom}(V,W)_0 := \{T : V \to W \mid T(V_i) \subset W_i,\ i \in \mathbb{Z}/2\mathbb{Z}\},$$
$$\mathrm{Hom}(V,W)_1 := \{T : V \to W \mid T(V_i) \subset W_{i+1},\ i \in \mathbb{Z}/2\mathbb{Z}\}.$$

For a field k with characteristic 0, $k^{p+q} =: k^{p,q}$ *can be attributed the structure of a vector superspace. Indeed, if $\{e_i\}_{i=1}^{p+q}$ is a basis for k^{p+q}, we classify $e_1,\dots,e_p$ as even and $e_{p+1},\dots,e_{p+q}$ as odd. This gives rise to the decomposition $k_0^{p,q} = k^p$ and $k_1^{p,q} = k^q$. Just as we can represent linear maps between vector spaces as matrices, we can view the endomorphisms in* $\mathrm{Hom}(k^{p,q},k^{p,q})$ *as matrices of the form*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} + \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$$

where A, B, C, and D are $p \times p, p \times q, q \times p$, and $q \times q$ matrices.

Let V and W be two vector superspaces. The *direct sum* $V \oplus W$ forms a vector superspace with grading given by $(V\oplus W)_0 = V_0 \oplus W_0$, and $(V \oplus W)_1 = V_1 \oplus W_1$. The *tensor product* $V \otimes W$ also forms a vector superspace with grading given by

$$(V\otimes W)_0 := (V_0 \otimes W_0) \oplus (V_1 \otimes W_1), \quad (V\otimes W)_1 := (V_0 \otimes W_1) \oplus (V_1 \otimes W_0).$$

A *scalar superproduct* is a non-degenerate, graded-symmetric, even, $\mathbb{R}$-bilinear form $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{R}$. In other words, for homogeneous $X,Y,Z \in V$, a scalar superproduct satisfies three conditions:

(i) $X \mapsto \langle X, \cdot\rangle$ is an isomorphism,
(ii) $\langle X,Y\rangle = (-1)^{|X||Y|}\langle Y,X\rangle$, and
(iii) $\langle aX+bY,Z\rangle = a\langle X,Z\rangle + b\langle Y,Z\rangle = (-1)^{|X+Y||Z|}\langle Z,aX+bY\rangle$.

We see that a scalar superproduct splits into a symmetric scalar product $\langle\cdot,\cdot\rangle_0$ on V_0 and a symplectic scalar product $\langle\cdot,\cdot\rangle_1$ on V_1. Hence, if there exists a scalar superproduct on V, the odd subspace V_1 must be even-dimensional.

Let $V^{m,n}$ be a vector superspace equipped with a scalar superproduct $\langle\cdot,\cdot\rangle$. A homogeneous basis $\{v_1,\cdots,v_{m+n}\}$ of V is called a $\langle\cdot,\cdot\rangle$*-normalised basis* if there exists $0 \le p \le m$ such that

$$(\langle v_i, v_j\rangle)_{i,j=1}^{m+n} = \begin{pmatrix} I_p & 0 & 0 & 0 \\ 0 & -I_{m-p} & 0 & 0 \\ 0 & 0 & 0 & I_r \\ 0 & 0 & -I_r & 0 \end{pmatrix}, \tag{2.1}$$

where I_k is the identity matrix of size k, and $2r = n$.

A *superalgebra* is a vector superspace $A = A_0 \oplus A_1$ that is a unital associative algebra whose multiplication respects the parity of the vector superspace, i.e., $A_iA_j \subset A_{i+j}$. We say a superalgebra is *supercommutative* if

$$ab = (-1)^{|a||b|}ba,$$

for homogeneous $a,b \in A$. Given a superalgebra A over a field k, let $D \in \mathrm{Hom}(A,A)$ be a k-linear map. We say that D is a *derivation* of A if

$$D(ab) = D(a)b + (-1)^{|D||a|}aD(b),$$

where $a,b \in A$. It becomes evident that a fundamental distinction between standard theory and graded theory arises in the factor that emerges when commuting two objects.

Let A be a superalgebra. We say a vector superspace M is a *left A-module* if there exists a morphism of supervector spaces $A \otimes M \to M$, given by $a \otimes m \mapsto am$, satisfying the following for $a,b \in A$ and $x,y \in M$:

(i) $a(x+y) = ax+ay$,
(ii) $(a+b)x = ax+bx$,
(iii) $(ab)x = a(bx)$, and
(iv) $1x = x$.

We say that an A-module M is *free* if it contains p even elements $\{e_1,\dots,e_p\}$ and q odd elements $\{\varepsilon_1,\dots,\varepsilon_q\}$ such that

$$M_0 = \mathrm{Span}_{A_0}\{e_1,\dots,e_p\} \oplus \mathrm{Span}_{A_1}\{\varepsilon_1,\dots,\varepsilon_q\},$$
$$M_1 = \mathrm{Span}_{A_1}\{e_1,\dots,e_p\} \oplus \mathrm{Span}_{A_0}\{\varepsilon_1,\dots,\varepsilon_q\}.$$

A free A-module is often denoted by $A^{p,q}$. Consider $T \in \mathrm{End}(A^{p,q})$. Referring to the discussion in example 2.1, we can represent T as a matrix of size $(p+q)\times(p+q)$ with block form

$$T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}.$$

Given that $A^{p,q}$ is a free A-module, entries in T_1 and T_4 have the same parity as T, while entries in T_2 and T_3 possess opposite parity. We define the *supertrace* of T to be $\mathrm{str}\, T = \mathrm{tr}\, T_1 - (-1)^T \,\mathrm{tr}\, T_4$. The supertrace is commutative:

$$\mathrm{str}(TS) = \mathrm{str}(ST).$$

2.2. Lie superalgebras and real forms

A *Lie superalgebra* is a vector superspace L endowed with an even morphism

$$[\cdot,\cdot] : L \otimes L \to L$$

such that for all homogeneous $x,y,z \in L$, $[\cdot,\cdot]$ satisfies graded symmetry and the graded Jacobi identity:

(i) $[x,y] + (-1)^{|x||y|}[y,x] = 0$, and
(ii) $[x,[y,z]] + (-1)^{|x|(|y|+|z|)}[y,[z,x]] + (-1)^{|y|(|z|+|x|)}[z,[x,y]] = 0$.

The set of all derivations of a superalgebra forms a Lie superalgebra.

EXAMPLE 2.2. *The superalgebra*

$$\mathrm{End}(V) := \mathrm{Hom}(V,V)_0 \oplus \mathrm{Hom}(V,V)_1$$

forms a Lie superalgebra with the supercommutator, which is defined for homogeneous elements as follows:

$$[X,Y] := XY - (-1)^{|X||Y|}YX.$$

This definition extends linearly to all elements of $\mathrm{End}(V)$*. If* $V = k^{m,n}$*, then* $\mathrm{End}(V)$ *with the supercommutator becomes the general linear superalgebra,* $\mathfrak{gl}(m|n)$*.*

Let $\mathfrak{g}$ be a Lie superalgebra. We define the *adjoint action* of $\mathfrak{g}$ on itself as the mapping $\mathrm{ad} : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$, given by $(X,Y) \mapsto [X,Y]$. For each $X \in \mathfrak{g}$, we obtain an endomorphism $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$, where $Y \mapsto [X,Y]$. Consequently, the adjoint action induces a representation of $\mathfrak{g}$, denoted as $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$. Analogous to the non-super case, the Jacobi identity is equivalent to the adjoint map being a derivation: $[\mathrm{ad}\ X, \mathrm{ad}\ Y] = \mathrm{ad}\ [X,Y]$.

Kac [Kac77] gives a classification of finite-dimensional simple complex Lie superalgebras. Consider $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_0^{\mathbb{C}} \oplus \mathfrak{g}_1^{\mathbb{C}}$ to be either $\mathfrak{gl}(m|n)^{\mathbb{C}}$ or a finite-dimensional basic classical Lie superalgebra over $\mathbb{C}$. That is, $\mathfrak{g}_0^{\mathbb{C}}$ is reductive (its adjoint representation is completely reducible) and $\mathfrak{g}^{\mathbb{C}}$ admits a non-degenerate invariant supersymetric even bilinear form $(\cdot,\cdot)$, which is $\mathrm{ad}_{\mathfrak{g}}$-invariant.

We say that a Lie subalgebra $\mathfrak{h}^{\mathbb{C}} \subset \mathfrak{g}_0^{\mathbb{C}}$ is a *Cartan subalgebra* of $\mathfrak{g}^{\mathbb{C}}$ if

(i) $\mathfrak{h}^{\mathbb{C}}$ is nilpotent, and
(ii) $\mathfrak{h}^{\mathbb{C}} = N_{\mathfrak{g}^{\mathbb{C}}}\mathfrak{h}^{\mathbb{C}} = \{X \in \mathfrak{g}^{\mathbb{C}} : [X,\mathfrak{h}] \subset \mathfrak{h}^{\mathbb{C}}\}$.

Let $(\mathfrak{h}^{\mathbb{C}})^*$ denote the dual space of $\mathfrak{h}^{\mathbb{C}}$. For $\alpha \in (\mathfrak{h}^{\mathbb{C}})^*$, define the vector superspace

$$\mathfrak{g}_\alpha^{\mathbb{C}} := \left\{ X \in \mathfrak{g}^{\mathbb{C}} : [H,X] = \alpha(H)X \text{ for all } H \in \mathfrak{h}^{\mathbb{C}} \right\}.$$

We say that $\alpha \in (\mathfrak{h}^{\mathbb{C}})^*$ is a root if $\alpha \neq 0$ and $\mathfrak{g}_\alpha^{\mathbb{C}} \neq \{0\}$. Denote by Δ the set of roots in $(\mathfrak{h}^{\mathbb{C}})^*$. We say that Δ is the *root system of* $\mathfrak{g}^{\mathbb{C}}$ *relative to* $\mathfrak{h}^{\mathbb{C}}$. A root α is *even* (resp. *odd*) if $\mathfrak{g}_\alpha^{\mathbb{C}} \subset \mathfrak{g}_0$ (resp. $\mathfrak{g}_\alpha^{\mathbb{C}} \subset \mathfrak{g}_1$). Denote by Δ_0 and Δ_1 the even and odd roots of $\mathfrak{g}^{\mathbb{C}}$ relative to $\mathfrak{h}^{\mathbb{C}}$, respectively. We have the *root space decomposition* of $\mathfrak{g}^{\mathbb{C}}$:

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha^{\mathbb{C}},$$

where $\mathfrak{g}_\alpha^{\mathbb{C}}$ are the *root subspaces* of $\mathfrak{g}^{\mathbb{C}}$. We say that $\Pi \subset \Delta$ is the set of *simple roots* if

(i) $\Pi = \{\alpha_1, \ldots, \alpha_l\}$, and
(ii) for each $\alpha \in \Delta$, there exist unique integers of the same sign $m_1, \ldots, m_l$ such that $\alpha = m_1\alpha_1 + \ldots + m_l\alpha_l$.

When $m_i \geq 0$ (resp. $m_i \leq 0$) for each $1 \leq i \leq l$, we say that α is a positive (resp. negative) root. The set of positive (resp. negative) roots is denoted Δ^+ (resp. Δ^-).

The bilinear form $(\cdot,\cdot)|_{\mathfrak{h}^{\mathbb{C}}}$ is non-degenerate, giving us a natural isomorphism between $\mathfrak{h}^{\mathbb{C}}$ and $(\mathfrak{h}^{\mathbb{C}})^*$. In other words, to each $\lambda \in (\mathfrak{h}^{\mathbb{C}})^*$, we associate the vector H_λ defined by $(H, H_\lambda) = \lambda(H)$ for all $H \in \mathfrak{h}^{\mathbb{C}}$. This allows us to define a real subalgebra of $\mathfrak{g}_0^{\mathbb{C}}$:

$$\mathfrak{h}_{\mathbb{R}} := \bigoplus_{\alpha \in \Delta} \mathrm{Span}_{\mathbb{R}}\{H_\alpha\}.$$

For each $\alpha \in \Delta$, let E_α denote a non-zero vector in $\mathfrak{g}_\alpha^{\mathbb{C}}$. It is easy to see that each $\mathfrak{g}_\alpha^{\mathbb{C}} = \mathrm{Span}_{\mathbb{C}}\{E_\alpha\}$ is one-dimensional. We can choose E_α in such a way that $B(E_\alpha, E_{-\alpha}) = -1$ and $[E_\alpha, E_{-\alpha}] = -H_\alpha$. We say that $\{H_\alpha, E_\alpha\}$ is a *Cartan-Weyl basis for* $\mathfrak{g}^{\mathbb{C}}$ *relative to* $\mathfrak{h}^{\mathbb{C}}$.

In order to study real Lie superalgebras, we consider the real forms of complex Lie superalgebras. We say that a real Lie superalgebra $\mathfrak{g}$ is a *real form* of a complex Lie superalgebra $\mathfrak{g}^{\mathbb{C}}$ if the complexification of $\mathfrak{g}$, $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$, is isomorphic to $\mathfrak{g}^{\mathbb{C}}$. We can characterise real forms of the above complex Lie superalgebras as follows. Define a transformation $* : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ such that, for all $X, Y \in \mathfrak{g}^{\mathbb{C}}$ and $z \in \mathbb{C}$,

$$[X,Y]^* = [Y^*, X^*], \quad (X^*)^* = X, \text{ and } (zX)^* = \bar{z}X^*.$$

The real form $\mathfrak{g}$ of $\mathfrak{g}^{\mathbb{C}}$ is then

$$\mathfrak{g} := \{X \in \mathfrak{g}_0^{\mathbb{C}} : X^* = -X\} \oplus \{\sqrt{-1}X \in \mathfrak{g}_1^{\mathbb{C}} : X^* = -X\}.$$

We say that $\mathfrak{g}$ is *compact* if $\mathfrak{g}_0$ is a compact real form of $\mathfrak{g}_0^{\mathbb{C}}$ in the classical sense. In a Cartan-Weyl basis, this becomes

$$\mathfrak{g} = \sqrt{-1}\mathfrak{h}_{\mathbb{R}} \oplus \bigoplus_{\alpha \in \Delta_0^+} \mathbb{R}A_\alpha \oplus \bigoplus_{\alpha \in \Delta_0^+} \mathbb{R}\sqrt{-1}B_\alpha \oplus \bigoplus_{\alpha \in \Delta_1^+} \mathbb{R}A_\alpha \oplus \bigoplus_{\alpha \in \Delta_1^+} \mathbb{R}\sqrt{-1}B_\alpha,$$

where $A_\alpha = E_\alpha - E_{-\alpha}$ and $B_\alpha = E_\alpha + E_{-\alpha}$.

EXAMPLE 2.3. *Consider the general linear superalgebra $\mathfrak{gl}(m|n)^{\mathbb{C}}$. If we fix a basis for $\mathbb{C}^{m,n}$, then*

$$\mathfrak{gl}(m|n)^{\mathbb{C}} = \mathfrak{gl}(m|n)_0^{\mathbb{C}} \oplus \mathfrak{gl}(m|n)_1^{\mathbb{C}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix} \right\},$$

where A, B, C, D are complex matrices of size $m \times m, m \times n, n \times m$, and $n \times n$, respectively. For $1 \leq i, j \leq m+n$, let E_{ij} denote the matrix with 1 in the $(i,j)^{th}$ entry and 0 elsewhere. By defining the transformation $(E_{ij})^* = E_{ji}$ for all $1 \leq i, j \leq m+n$, we obtain the compact real form $\mathfrak{u}(m|n)$.

2.3. Prelude into the theory of sheaves

One approach to defining a supermanifold follows the familiar definition of a manifold. This involves glueing together patches that locally resemble a model space with both commuting and anti-commuting coordinates. The contribution of anti-commuting variables is nuanced because of their nilpotency. Kostant [Kos77] was among the pioneering mathematicians who recognised the analogy between supermanifolds and schemes – spaces endowed with a family of rings containing nilpotent elements. As a result, using Grothendieck's theory of schemes, a second definition of a supermanifold was introduced. Before delving into this definition of a supermanifold, it is crucial to establish some foundational concepts from algebraic geometry. For a comprehensive exposition of this elegant theory, we suggest that readers consult the books [EH00] and [Har77].

Sheaves are fundamental to algebraic geometry because they provide a way to track local algebraic information on a topological space. In this section, we present the definition of a sheaf and some related terminology. We introduce an alternate method for defining a smooth manifold that leverages this abstract language. This approach serves as a guide for the subsequent discussion of supermanifolds.

A *sheaf* of commutative algebras $\mathscr{F}$ on a topological space M is a map that assigns a commutative algebra $\mathscr{F}(V)$ to each open set $V \subset M$ such that for every $U \subset V$, there exists a restriction morphism $r_{V,U} : \mathscr{F}(V) \to \mathscr{F}(U)$ (sometimes denoted by $\cdot|_U$), satisfying the following properties:

(i) there exists an identity map $r_{U,U} = \mathrm{id}_M$;
(ii) for any open covering $\{U_i\}_{i\in I}$ of U and a family $\{f_i\}_{i\in I}$, $f_i \in \mathscr{F}(U_i)$ satisfying $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$ for all $i, j \in I$, then there exists a unique $f \in \mathscr{F}(U)$ with $f|_{U_i} = f_i$.

The second condition is known as the *glueing property*. If only (i) is satisfied, we call $\mathscr{F}$ a *presheaf*. The elements in $\mathscr{F}(U)$ are called *sections* over U; when $U = M$, we call such elements *global sections*.

EXAMPLE 2.4. *The following are examples of sheaves:*

(i) Given any topological space X, one may define the sheaf of continuous real-valued functions on X.
(ii) Given a differentiable manifold, we may consider the commutative algebra of smooth function $\mathscr{C}^\infty(U)$ defined on some open subset U. Taking the restriction map to be the restriction of smooth functions, $\mathscr{C}^\infty$ forms a sheaf.
(iii) Let X be a topological space. Define an algebra of functions on $U \subset X$ by $\mathscr{O}(U) := \{f : U \to \mathbb{R} \mid f \text{ is constant}\}$. This assignment forms a presheaf but not a sheaf. To see this, let $U = U_1 \cup U_2$, and define $f_1 \in \mathscr{O}(U_1)$ by $f_1(x_1) = 1$, and $f_2 \in \mathscr{O}(U_2)$ by $f_2(x_2) = 2$. We find that there exists no constant function $f \in \mathscr{O}(U)$ satisfying $f|_{U_i} = f_i$.

Let $\mathscr{F}$ be a presheaf on a topological space M, and fix a point $x \in M$. For open neighbourhoods, U and V of x, let s and t be sections over U and V, respectively. We say that (U,s) and (V,t) are equivalent if there exists a neighbourhood $W \subset U \cap V$ such that $s|_W = t|_W$.The set of all such equivalent pairs forms the *stalk at* x, denoted by $\mathscr{F}_x$. The elements of $\mathscr{F}_x$ are called *germs*.

Let $\mathscr{F}$ and $\mathscr{G}$ be presheaves on M. A *morphism of presheaves* $\phi : \mathscr{F} \to \mathscr{G}$ is a family of algebra morphisms $\phi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ defined for each open $U \subset M$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathscr{F}(V) & \xrightarrow{\phi_V} & \mathscr{G}(V) \\ {\scriptstyle r_{V,U}}\downarrow & & \downarrow{\scriptstyle r_{V,U}} \\ \mathscr{F}(V) & \xrightarrow{\phi_U} & \mathscr{G}(U) \end{array}$$

A *morphism of sheaves* is a morphism of the underlying presheaves.

REMARK 2.5. *Any morphism of presheaves $\phi : \mathscr{F} \to \mathscr{G}$ induces a morphism on the stalks $\phi_x : \mathscr{F}_x \to \mathscr{G}_x$.*

If all the induced stalk morphisms injective, then a morphism of sheaves is considered *injective*. Surjectivity can be defined in a similar manner, but one should be cautious as it is possible to have a surjective sheaf morphism $\phi : \mathscr{F} \to \mathscr{G}$ where ϕ_U is not surjective for some U.

Suppose we have a morphis of sheaves, $\phi : \mathscr{F} \to \mathscr{G}$. By mapping U to the kernel of $\phi(U)$, we can always define a sheaf. However, mapping U to the image of $\phi(U)$ only defines a presheaf in general. When we have a presheaf, we naturally wonder if it can be turned into a sheaf. The following explains how this can be accomplished.

Let $\mathscr{F}$ be a presheaf on M. The *sheafification* of $\mathscr{F}$ is the unique sheaf $\widetilde{\mathscr{F}}$ and a morphism of presheaves $\varphi : \mathscr{F} \to \widetilde{\mathscr{F}}$ such that for all $x \in M$, $\varphi_x : \mathscr{F}_x \to \widetilde{\mathscr{F}_x}$ is an isomorphism.

Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves on a topological space M, and define an injective morphism of sheaves such that $\mathscr{G}(U) \subset \mathscr{F}(U)$ for all $U \subset M$. We can then consider the *quotient sheaf* $\mathscr{F}/\mathscr{G}$ to be the sheafification of the image presheaf $U \mapsto \mathscr{F}(U)/\mathscr{G}(U)$.

If M is a topological space and $\mathscr{F}$ is a sheaf of commutative rings on M, we call the pair $\mathscr{M} = (M, \mathscr{F})$ a *ringed space*. If, in addition, each stalk $\mathscr{F}_x$ has a unique maximal ideal, i.e., it is a local ring, we say that $\mathscr{M}$ is a *locally ringed space*.

A morphism of ringed spaces $\Phi : \mathscr{M} \to \mathscr{N}$ is a pair $\Phi = (\phi, \phi^*)$ where $\phi : M \to N$ is a topological space morphism and $\phi^* : \mathscr{G} \to \phi_* \mathscr{F}$ is a sheaf morphism. Here $\phi_* \mathscr{F}$ is a sheaf defined on N by $(\phi_* \mathscr{F})(U) = \mathscr{F}(\phi^{-1}(U))$ for every open $U \subset N$.

Every morphism of ringed spaces induces a morphism on the stalks, $\phi_x : \mathscr{G}_{\phi(x)} \to \mathscr{F}_x$. Let $\mathfrak{m}_{M,x}$ and $\mathfrak{m}_{N,\phi(x)}$ be the maximal ideals of the stalks $\mathscr{F}_x$ and $\mathscr{G}_{\phi(x)}$, respectively. If Φ satisfies $\phi_x^{-1}(\mathfrak{m}_{M,x}) = \mathfrak{m}_{N,\phi(x)}$, we say Φ is a *locally ringed space morphism*.

To end this section, we present an alternate definition of a smooth manifold, adopting the abstract language of sheaves.

Let M be a Hausdorff, second countable topological space and $\mathscr{O}_M$ be a sheaf of commutative algebras on M such that $(M, \mathscr{O}_M)$ is a locally ringed space. We say $(M, \mathscr{O}_M)$ is a *smooth manifold* of dimension n if it is isomorphic as a locally ringed space to $(\mathbb{R}^n, \mathscr{C}^\infty_{\mathbb{R}^n})$, where $\mathscr{C}^\infty_{\mathbb{R}^n}$ is the sheaf of smooth functions on $\mathbb{R}^n$.

We must check that this definition agrees with the classical construction of a smooth manifold. This fact requires three key ideas:

1. the usual construction of a manifold M gives rise to a sheaf $\mathscr{O}$ such that $(M, \mathscr{O})$ is a ringed space;
2. the ringed space $(M, \mathscr{O})$ is locally isomorphic to $(\mathbb{R}^n, \mathscr{C}^\infty_{\mathbb{R}^n})$;
3. given a ringed space $(M, \mathscr{O}_M)$ that is locally isomorphic to $(\mathbb{R}^n, \mathscr{C}^\infty_{\mathbb{R}^n})$, we can construct an atlas on M.

REMARK 2.6. *With notation as in* (3) *and* (1)*, the sheaf* $\mathscr{O}$ *turns out to be isomorphic to the sheaf* $\mathscr{O}_M$.

LEMMA 2.7. *Let* M *be a Hausdorff, second countable topological space equipped with a maximal smooth atlas* $\mathscr{A} := \{(U_i, \phi_i)\}$*. Then, there exists a locally ringed space* $(M, \mathscr{C}_M^\infty)$ *where* $\mathscr{C}_M^\infty$ *is the sheaf of smooth functions on* M.

PROOF. For each open subset $U \subset M$, consider the assignment $U \mapsto \mathscr{C}_M^\infty(U)$, where $\mathscr{C}_M^\infty(U)$ is the algebra of smooth functions on U. We define the restriction morphisms by taking the usual restriction of functions: $r_{V,U}(f) = f|_U$ for open sets $U \subset V \subset M$. It follows that $\mathscr{C}_M^\infty$ is a presheaf. To see that it is in fact a sheaf, take an open cover $\{U_i\}_{i\in I}$ of U and a family of smooth functions $\{f_i\}_{i\in I}$ such that f_i and f_j agree on the intersection $U_i \cap U_j$, for all $i \neq j$. The topological pasting lemma asserts the existence of a unique continuous function f such that $f|_{U_i} = f_i$. This f is smooth since it restricts to smooth functions on U.

We know $(M, \mathscr{C}_M^\infty)$ is a ringed space by definition. For each $x \in M$, we define a mapping from the stalk $\mathscr{C}_{M,x}^\infty \to \mathbb{R}$ by $[(U, f)] \mapsto f(x)$. Due to the constant functions being smooth, this mapping is surjective. We know that the kernel K of this map is an ideal, and so we may consider the quotient of rings $\mathscr{C}_{M,x}^\infty/K$. By the first isomorphism theorem for rings, $\mathscr{C}_{M,x}^\infty/K \cong \mathbb{R}$. Since the above quotient ring is isomorphic to a field, the ideal K must be maximal. It is readily seen that any element not in K must be invertible and so it is the unique maximal ideal. This shows that $(M, \mathscr{C}_M^\infty)$ is a locally ringed space. □

LEMMA 2.8. $(M, \mathscr{C}_M^\infty)$ *is locally isomorphic as a locally ringed space to* $(\mathbb{R}^n, \mathscr{C}_{\mathbb{R}^n}^\infty)$.

PROOF. We aim to demonstrate that for every $x \in M$, there exists an open neighbourhood $U_i \subset M$ of x, such that $\Phi : (U, \mathscr{C}_U^\infty) \to (V, \mathscr{C}_V^\infty)$ is an isomorphism of ringed spaces. Here, $\mathscr{C}_U^\infty$ denotes the sheaf of smooth functions restricted to the subset $U \subset M$. In essence, we must identify, for each $x \in M$, a homeomorphism $\varphi : U \to V$ and an isomorphism of sheaves $\Phi^* : \mathscr{C}_V^\infty \to \Phi_* \mathscr{C}^\infty$, where $(\Phi_* \mathscr{C}_U^\infty)(W) := \mathscr{C}_U^\infty(\varphi(W))$ for some open $W \subset V$.

In fact, for any $x \in M$, there exists an open subset $U_i \subset M$ containing x. Furthermore, we have the homeomorphism $\phi_i : U_i \to \phi_i(U_i) =: V_i$. We proceed to define the sheaf isomorphism Φ^* using its underlying ring isomorphisms, $\Phi_{V_i}^* : \mathscr{C}_{V_i}^\infty(W) \to \mathscr{C}_{U_i}^\infty(\phi_i^{-1}(W))$ where $W \subset V_i$. For a smooth function f defined on $V_i \subset \mathbb{R}^n$, we define $\Phi_{V_i}^*$ by $f \mapsto f \circ \phi_i$. This clearly defines a ring isomorphism, thereby defining a sheaf isomorphism Φ^*.

Given that we are working with locally ringed spaces, each stalk is a local ring. Consequently, our ring isomorphisms $\Phi_{V_i}^*$ map units to units, preserving the local ring property. □

THEOREM 2.9. *Let* M *be a Hausdorff, second countable topological space and* $\mathscr{O}_M$ *a sheaf of commutative algebras such that* $(M, \mathscr{O}_M)$ *is a locally ringed space isomorphic to* $(\mathbb{R}^n, \mathscr{C}_{\mathbb{R}^n}^\infty)$*.Then, there exists a maximal smooth atlas on* M *such that* $(M, \mathscr{C}_M^\infty) \cong (M, \mathscr{O}_M)$*, where* $\mathscr{C}_M^\infty$ *is the sheaf of smooth functions on* M.

PROOF. We are given a local isomorphism between $(M, \mathscr{O}_M)$ and $(\mathbb{R}^n, \mathscr{C}_{\mathbb{R}^n}^\infty)$. As a result, for each $x \in M$, there exists a neighbourhood $U_i \subset M$ containing x and an isomorphism $\Phi : (U_i, \mathscr{O}_{U_i}) \to (V_i, \mathscr{C}_{V_i}^\infty)$, consisting of a homeomorphism $\phi_i : U_i \to V_i$ and a sheaf isomorphism $\Phi^* : \mathscr{C}_{V_i}^\infty \to \Phi_* \mathscr{O}_{U_i}$. We can construct an atlas from the topological maps on the open subsets of M, $\{(U_i, \phi_i)\}$. It remains to show that the transition maps are smooth.

Consider two open subsets U and V of M, and their respective locally ringed space isomorphisms, $\Phi_1 = (\varphi_1, \Phi_1^*)$ and $\Phi_2 = (\varphi_2, \Phi_2^*)$. Consider the restricted isomorphisms

$$\Phi_1 : (U \cap V, \mathscr{O}_{U\cap V}) \to \left(\varphi_1(U \cap V), \mathscr{C}_{\varphi_1(U\cap V)}^\infty\right), \text{ and}$$

$$\Phi_2 : (U \cap V, \mathscr{O}_{U\cap V}) \to \left(\varphi_2(U \cap V), \mathscr{C}^\infty_{\varphi_2(U\cap V)}\right).$$

It is evident that $\Phi_1 \circ \Phi_2^{-1}$ is an isomorphism. Define open sets $W \subset \varphi_1(U \cap V)$ and $\widetilde{W} = \left(\varphi_1 \circ \varphi_2^{-1}\right)^{-1}(W)$. Then, the ring morphism $(\Phi_1 \circ \Phi_2^{-1})^*_W : \mathscr{C}^\infty(W) \to \mathscr{C}^\infty(\widetilde{W})$ is given by $f \mapsto f \circ \varphi_1 \circ \varphi_2^{-1}$. To see this, assume the contrary. Without loss of generality, we can translate f such that $(\Phi_1 \circ \Phi_2^{-1})^*_W(f)|_x = 0$ and $f \circ \varphi_1 \circ \varphi_2^{-1}(x) \neq 0$. In other words, $[f] \in \mathscr{C}^\infty_{\varphi_1\circ\varphi_2^{-1}(x)}$ is a non-zero, invertible germ and $[(\Phi_1 \circ \Phi_2^{-1})^*_W(f)] \in \mathscr{C}^\infty_x$ is the zero germ. This is a contradiction as ring isomorphisms map units to units.

Finally, we have that $f \circ \varphi_1 \circ \varphi_2^{-1} : \widetilde{W} \to W \to \mathbb{R}$ is smooth for all $f : W \to \mathbb{R}$, and so $\varphi_1 \circ \varphi_2^{-1}$ is smooth. This shows that transition maps in our atlas are smooth. Thus, $(M, \mathscr{O}_M)$ has the structure of a smooth manifold. $\square$

2.4. Supermanifolds – the algebro-geometric construction

In this section, we introduce the concept of a supermanifold from an algebro-geometric perspective. This elegant formulation enables us to extend most concepts from differential geometry.

Following the alternative definition of a smooth manifold presented earlier, we introduce the graded analogue of ringed spaces, which paves the way for defining a supermanifold. A *superringed space*, denoted as $\mathscr{S}$, refers to a topological space S equipped with a sheaf of supercommutative rings, $\mathscr{O}_S$. We say a superringed space is a *superspace* if each stalk admits a unique homogeneous maximal ideal $I = (I \cap R_0) \oplus (I \cap R_1)$. In such instances, we say each stalk is a *local superring*.

Just as we have defined morphisms for ringed spaces and locally ringed spaces, we establish analogous definitions for their super counterparts. It is important to note that in the super setting, morphisms must preserve the parity of elements.

EXAMPLE 2.10. *Let M be a smooth manifold with the sheaf of smooth functions $\mathscr{C}^\infty_M$. We can define the sheaf of supercommutative $\mathbb{R}$-algebras by the assignment*

$$U \mapsto \mathscr{O}_M(U) := \mathscr{C}^\infty_M(U) \otimes \Lambda(\xi_1, \ldots, \xi_n),$$

where U is an open subset of M. In a sense, we can view elements of these rings as superfunctions:

$$f(x,\xi) = \sum_\mu f_\mu(x)\xi^\mu = \sum_\mu f_\mu(x)\xi_1^{\mu_1} \wedge \ldots \wedge \xi_n^{\mu_n}, \tag{2.2}$$

where $f_\mu(x_1, \ldots, x_m) \in C^\infty_M(U)$ and $\mu \in (\mathbb{Z}/2\mathbb{Z})^n$.

Equipping M with this sheaf gives rise to a superspace $(M, \mathscr{O}_M)$, where the maximal ideal of $\mathscr{O}_{M,x}$ is generated by the maximal ideal of $\mathscr{C}^\infty_{M,x}$ and the odd indeterminates $\xi_1, \ldots, \xi_n$. In particular, when $M = \mathbb{R}^m$, we obtain flat superspace

$$\mathbb{R}^{m,n} := (\mathbb{R}^m, \mathscr{C}^\infty_{\mathbb{R}^m} \otimes \Lambda(\xi_1, \ldots, \xi_n)).$$

This serves as the model space for our definition of a supermanifold.

Let $(S, \mathscr{O}_S)$ be a superspace and take an open subset U of S. Then, there exists a superspace $(U, \mathscr{O}_S|_U)$ known as the *open subsuperspace* associated with U (although we often just say subspace). The subsequent example outlines the general linear supergroup as a subspace of the matrix group – an extremely significant space in the progression of supergeometry.

EXAMPLE 2.11. *Let* $\mathsf{M}(m|n) = \mathbb{R}^{m^2+n^2,2mn}$ *denote the superspace corresponding to the vector space of* $((m+n)\times(m+n))$*-matrices. As a vector superspace,* $\mathsf{M}(m|n)$ *decomposes into*

$$(\mathsf{M}(m|n))_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \right\}, \text{ and } (\mathsf{M}(m|n))_1 = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \right\},$$

where A,B,C,D *are* $(m\times m),(m\times n),(n\times m),(n\times n)$*-matrices respectively. Define* m^2+n^2 *even coordinates* t_{ij} *corresponding to the matrices A and D, where* $i,j\in\{1,\dots,m\}$ *or* $i,j\in\{m+1,\dots,m+n\}$*. We also define* $2mn$ *odd coordinates* ξ_{kl} *corresponding to the matrices B and C, where* $k\in\{1,\dots,m\}$ *and* $l\in\{m+1,\dots,m+n\}$ *or* $k\in\{m+1,\dots,m+n\}$ *and* $l\in\{1,\dots,n\}$*.*

The structure sheaf of $\mathsf{M}(m|n)$ *is defined by the assignment*

$$V \mapsto \mathscr{O}_{\mathsf{M}(m|n)}(V) := \mathscr{C}^\infty_{\mathsf{M}_m\times\mathsf{M}_n}(V)\otimes\Lambda(\xi_{kl})$$

for all open $V\subset\mathsf{M}_m\times\mathsf{M}_n$*. Consider the open set* $U\subset\mathsf{M}_m\times\mathsf{M}_n$ *which has* $\det\left(t^{ij}\right)\neq 0$ *for* $i,j\in\{1,\dots,m\}$ *or* $i,j\in\{m+1,\dots,m+n\}$*. Define the open subspace of* $\mathsf{M}(m|n)$ *associated with the open set U to be the general linear supergroup* $\mathsf{GL}(m|n) := \left(U,\mathscr{O}_{\mathsf{M}(m|n)}|_U\right)$*.*

REMARK 2.12. *We often omit the field over which we are working. Unless otherwise stated, we will only consider* $\mathbb{R}$*.*

Let $\mathscr{C}^\infty_U$ be the sheaf of smooth functions on the domain $U\subset\mathbb{R}^m$. We call the superspace $U^{m,n} = (U,\mathscr{C}^\infty_{\mathbb{R}^m}|_U\otimes\Lambda(\xi_1,\dots,\xi_n))$ a *superdomain*. We are finally ready to define a supermanifold.

A superspace $\mathscr{M} = (M,\mathscr{O}_M)$ is called a *supermanifold* if the following two conditions hold:

(i) M is a locally compact, second countable, Hausdorff topological space;
(ii) for each $x\in M$, there exists an open neighbourhood U containing x such that there is an isomorphism of superringed spaces

$$(U,\mathscr{O}_{M|_U}) \to U^{m,n}\subset\mathbb{R}^{m,n},$$

where $U^{m,n}$ is a superdomain of $\mathbb{R}^{m,n}$.

It isn't clear what it means to evaluate a superfunction at a point, or even what points are in $\mathbb{R}^{m,n}$. The evaluation of f at a point $(x_1,\dots,x_m)\in U$ gives a value $f(x;\xi)\in\mathbb{R}\otimes\Lambda(\xi_1,\dots,\xi_n)$. It is shown in [Var04] that an element $s=\sum_\mu s_\mu\xi^\mu\in R\otimes\Lambda(\xi_1,\dots,\xi_n)$ is invertible if and only if s_0 is invertible within a unital commutative ring R.

Applying this result to $R=\mathscr{C}^\infty_M(U)$, we deduce that a superfunction of the form (2.2) is invertible if and only if f_0 is invertible in $\mathscr{C}^\infty_M(U)$. Consequently, we define the value of a superfunction $f\in\mathscr{C}^\infty_M(U)\otimes\Lambda(\xi_1,\dots,\xi_n)$ at a point $x\in U$ to be the unique value $k\in\mathbb{R}$ such that $f-k$ is not invertible in any neighbourhood of $x\in U$.

EXAMPLE 2.13. *Take* $M=\mathbb{R}^{1,1}$*, with global coordinates* $(t;\xi)$*. The global section* $f=t\xi\in\mathscr{O}_M(\mathbb{R})$ *has the property that for any non-zero* $c\in\mathbb{R}$*,* $t\xi-c$ *is invertible. Indeed, the inverse is given by* $-c^{-2}t\xi-c^{-1}$*; hence, the value of f at every point in* $\mathbb{R}^{1,1}$ *is* 0*.*

For every supermanifold, we can establish an underlying smooth manifold by factoring out the nilpotent elements. Given a supermanifold $\mathscr{M}=(M,\mathscr{O}_M)$ and an open subset $U\subset M$, there exists a map $\varepsilon_U:\mathscr{O}_M(U)\to\mathscr{F}(U)$ defined by $f\mapsto\tilde{f}$, where $\tilde{f}:U\to\mathbb{R}$ is the evaluation map. The image of ε_U, which we denote by $\mathscr{F}(U)$, is an algebra, thereby making ε_U a surjective algebra morphism. This gives rise to the short exact sequence

$$0\to\mathscr{J}(U)\to\mathscr{O}_M(U)\to\mathscr{F}(U)\to 0,$$

where $\mathscr{J}(U) := \ker \varepsilon_U$. It can be demonstrated that the assignment $U \mapsto \mathscr{J}(U)$ defines a sheaf. For each $U \subset M$, exactness results in the isomorphism $\mathscr{F}(U) \cong \mathscr{O}_M(U)/\mathscr{J}(U)$. As the quotient of two sheaves defines only a presheaf in general, we let $\mathscr{F}$ denote the sheafification of the presheaf defined by $U \mapsto \mathscr{O}_M(U)/\mathscr{J}(U)$.

This process constructs $\mathscr{F}$ to be locally isomorphic to $\mathscr{C}^\infty_{\mathbb{R}^m}$, and so the ringed space $M_{\text{Red}} := (M, \mathscr{F})$ is locally isomorphic to $(\mathbb{R}^m, \mathscr{C}^\infty_{\mathbb{R}^m})$. We call M_{Red} the *body manifold* of $\mathscr{M}$.

The body manifold not only gives an intuitive perspective on supermanifolds, but also enables us to formulate a system of local coordinates on $\mathscr{M}$. Specifically, if $U \subset M_{\text{Red}}$ is such that $\mathscr{O}_M(U) \cong \mathscr{C}^\infty_{\mathbb{R}^m}(U) \otimes \Lambda(\xi_1, \dots, \xi_n)$, we define $(x_i; \xi_j)$ to be *coordinates* of $\mathscr{M}$ on U, where $\{x_i\}_{i=1}^m$ denote the standard coordinates on $U \subset M_{\text{Red}}$.

We end this section by offering an alternative approach to understanding functions on supermanifolds.

Let $\mathscr{M} = (M, \mathscr{O}_M)$ and $\mathscr{V}^{p,q} = (V, \mathscr{C}^\infty_{\mathbb{R}^p} \otimes \Lambda(\eta_1, \dots, \eta_q))$ be supermanifolds. Define $\Psi := (\psi, \psi^*) : \mathscr{M} \to \mathscr{V}^{p,q}$ to be a morphism of supermanifolds. Denote the global system of coordinates on $\mathscr{V}^{p,q}$ by $(y_i; \eta_j)$. The functions

$$f_i = \psi^* y_i, \quad 1 \le i \le p, \text{ and } \theta_j = \psi^* \eta_j, \quad 1 \le j \le q$$

are such that

(i) $f_i \in \mathscr{O}_{M,0}(M)$,
(ii) $\theta_j \in \mathscr{O}_{M,1}(M)$, and
(iii) $(\varepsilon f_1, \dots, \varepsilon f_p)(M) \subset V$.

The pullbacks of these super coordinate functions completely determine the morphism.

THEOREM 2.14 (Global Chart Theorem). *If $\mathscr{M} = (M, \mathscr{O}_M)$ is a supermanifold, $V \subset \mathbb{R}^p$ an open subset, and $(f_i; \theta_j)$ is a $(p+q)$-tuple of global sections in $\mathscr{O}_M(M)$ that satisfy the above two conditions, then there exists a unique morphism of supermanifolds $\Psi = (\psi, \psi*) : \mathscr{M} \to \mathscr{V}^{p,q}$ such that $f_i = \psi^* y_i$, and $\theta_j = \psi^* \eta_j$.*

PROOF. This proof is quite technical, so we refer the reader to [CCF11, Theorem 4.2.5]. □

EXAMPLE 2.15. *Consider the supermanifold $\mathbb{R}^{1,2}$ and a morphism $\Psi : \mathbb{R}^{1,2} \to \mathbb{R}^{1,2}$. In the global coordinates $\{x, \xi_1, \xi_2\}$, a section f can be written as*

$$f(x, \xi_1, \xi_2) = f_0(t) + f_1(x)\xi_1 + f_2(x)\xi_2 + f_{12}(x)\xi_1\xi_2.$$

Recall that Ψ consists of a topological map ψ and a sheaf morphism, ψ^*. Let us prescribe the images of the global coordinates under ψ^*:

$$x \mapsto x^* = x + \xi_1\xi_2, \quad \xi_1 \mapsto \xi_1^* = \xi^1, \quad \xi_2 \mapsto \xi_2^* = \xi_2.$$

Under this mapping, we see f map to

$$\psi^*(f) = f(x^*, \xi_1^*, \xi_2^*) = f_0(x^*) + f_1(x^*)\xi_1 + f_2(x^*)\xi_2 + f_{12}(x^*)\xi_1\xi_2.$$

To make sense of this expression, we Taylor expand $f_I(x^*) = f_I(x + \xi_1\xi_2) = f_I(x) + \xi_1\xi_2 f_I'(x)$ for each index I. This expansion terminates due to the nilpotency of the odd coordinates. Hence, defining the image of global sections completely determines the supermanifold morphism.

2.5. The tangent sheaf and vector fields

In this section, we consider a supermanifold $\mathscr{M} = (M, \mathscr{O}_M)$ and define analogues for the tangent space and tangent bundle. This will provide the groundwork to study the local structure of maps between supermanifolds.

For each $U \subset M$, let $\mathrm{Der}(\mathscr{O}_M(U))$ be the set of superderivations of $\mathscr{O}_M(U)$. Recall that an element $D \in \mathrm{Der}(\mathscr{O}_M(U))$ is a map $D : \mathscr{O}_M(U) \to \mathbb{R}$ such that

$$D(st) = (Ds)t + (-1)^{|D||s|}s(Dt)$$

for $s,t \in \mathscr{O}_M(U)$. The set $\mathrm{Der}(\mathscr{O}_M(U))$ is a vector superspace and has a super $\mathscr{O}_M(U)$-module structure given by

$$(sD)t := s(Dt), \quad sD \in \mathrm{Der}_{|s|+|D|}(\mathscr{O}_M(U)).$$

Define the *tangent bundle* of $\mathscr{M}$ to be the $\mathscr{O}_M(U)$-module of derivations of $\mathscr{O}_M(U)$, denoted by $\mathscr{T}_M(U) := \mathrm{Der}(\mathscr{O}_M(U))$. We have a natural restriction map $\mathscr{T}_M(V) \to \mathscr{T}_M(U)$ for $U \subset V \subset M$, turning $\mathscr{T}_M$ into a sheaf of $\mathscr{O}_M$-modules. Due to its sheaf structure, we sometimes refer to $\mathscr{T}_M$ as the *tangent sheaf* of $\mathscr{M}$.

The sections of $\mathscr{T}_M$ are called *vector fields* on M. Concretely, a vector field X on $\mathscr{M}$ is a family of super derivations $X_U : \mathscr{O}_M(U) \to \mathscr{O}_M(U)$ that is compatible with restrictions.

For each $U \subset M$, we furnish $\mathscr{T}_M(U)$ with a bracket operation, defined by $[X,Y]f := X(Yf) - (-1)^{|X||Y|}Y(Xf)$ for all vector fields X and Y, and superfunctions f. This satisfies the graded Jacobi identity, allowing us to view $\mathscr{T}_M(U)$ as a Lie superalgebra.

Given a system of local coordinates $(x;\xi)$ on $U \subset \mathscr{M}$, we have basic even and odd vector fields $\frac{\partial}{\partial x_i} \in \mathrm{Der}_0\mathscr{O}_M(U)$ and $\frac{\partial}{\partial \xi_j} \in \mathrm{Der}_1\mathscr{O}_M(U)$ which act on superfunctions by

$$\frac{\partial}{\partial x_i} f = \sum_\mu \frac{\partial f_\mu(x_1,\ldots,x_m)}{\partial x_i}\xi^\mu,$$
$$\frac{\partial}{\partial \xi_j} f = \sum_\mu \mu_j(-1)^{\mu_1+\cdots+\mu_{j-1}} f_\mu(x_1,\ldots,x_m)\xi_1^{\mu_1}\ldots\xi_{j-1}^{\mu_{j-1}}\xi_{j+1}^{\mu_{j+1}}\ldots\xi_n^{\mu_n}.$$

Analogously to the non-super setting, every supervector field admits a local description:

$$X = \sum_{i=1}^m \mathscr{X}_i\frac{\partial}{\partial x_i} + \sum_{j=1}^n \mathscr{X}_j\frac{\partial}{\partial \xi_j}, \tag{2.3}$$

where $\mathscr{X}_i, \mathscr{X}_j \in \mathscr{O}(U)$. Naturally, for every point $p \in M$, the *tangent space* $T_pM = T_pM_0 \oplus T_pM_1$ of $\mathscr{M}$ at p is defined to be the space of superderivations $\varphi : \mathscr{O}_{M,p} \to \mathbb{R}$. For each $p \in U \subset M$, there is a natural mapping $\mathscr{T}_M(U) \to T_pM$ sending a vector field X to its value X_p. It is important to note that unlike in the non-super theory, a supervector field is not determined by its value at all points.

Let $\mathscr{M}$ and $\mathscr{N}$ be supermanifolds. Recall that a morphism of superringed spaces $\Phi : \mathscr{M} \to \mathscr{N}$ is a pair $\Phi = (\phi,\phi^*)$ consisting of a topological map and a morphism of sheaves. Unlike the scenario with a morphism of ringed spaces, a morphism of superringed spaces isn't determined by the topological map alone. In the super context, the map $\phi^* : \mathscr{O}_N(N) \to \phi_*\mathscr{O}_M(M) = \mathscr{O}(\phi^{-1}(M))$ acting on global sections determines the morphism Φ.

Consider a morphism of supermanifolds $\Phi : \mathscr{M} \to \mathscr{N}$ and a subset $V \subset N$. Define a *vector field along a morphism* to be a morphism of supervector spaces

$$Y : \mathscr{O}_N(V) \to \phi_*\mathscr{O}_M(V) = \mathscr{O}_M(\phi^{-1}(V))$$

such that the homogeneous components of Y satisfy

$$Y(fg) = (Yf)\cdot\phi^*(g) + (-1)^{|Y||f|}\phi^*(f)\cdot(Yg)$$

for all $f,g \in \mathscr{O}_N(V)$. The set of such vector fields defines a sheaf, which we denote by $\mathscr{T}_\Phi$. Any $Y \in \mathscr{T}_\Phi(V)$ can be uniquely written in local coordinates as

$$Y = \sum_{i=1}^{m} \mathscr{Y}_i \cdot \left(\phi^* \circ \frac{\partial}{\partial x_i}\right) + \sum_{j=1}^{n} \mathscr{Y}_j \cdot \left(\phi^* \circ \frac{\partial}{\partial \xi_j}\right), \tag{2.4}$$

where $\mathscr{Y}_i, \mathscr{Y}_j \in \phi_* \mathscr{O}_M(V) = \mathscr{O}_N(\phi^{-1}(V))$.

Given a supermanifold morphism $\Phi : \mathscr{M} \to \mathscr{N}$, the *differential* or *pushforward* $\mathrm{d}\Phi : \mathscr{T}_M(M) \to \mathscr{T}_\Phi(N)$ is defined by the mapping $X \mapsto \mathrm{d}\Phi(X) := X \circ \phi^*$. Locally, the differential of Φ at $p \in M$ is given by $v \mapsto v \circ \phi^*$ and denoted $\mathrm{d}\Phi_p : T_pM \to T_{\phi(p)}N$. Specifically, given a local coordinate system $\{x;\xi\}$ on $U \subset M$ and a vector field $X \in \mathscr{T}_M(U)$, we compute

$$\mathrm{d}\Phi(X) = X \circ \phi^* = \sum_i X(\phi^* x_i) \cdot \phi^* \circ \frac{\partial}{\partial x_i} + \sum_j X(\phi^* \xi_j) \cdot \phi^* \circ \frac{\partial}{\partial \xi_j}.$$

We say that a supermanifold morphism $\Phi : \mathscr{M} \to \mathscr{N}$ is a *local superdiffeomorphism* at a point $p \in M$ if $\mathrm{d}\Phi_p$ is bijective. If Φ is a local superdiffeomorphism for all $p \in M$ and its inverse is a supermanifold morphism, then Φ is a *superdiffeomorphism*.

THEOREM 2.16 (Inverse Function Theorem). *Let $\Phi : \mathscr{M} \to \mathscr{M}$ be a morphism of supermanifolds, and let $p \in M$ be such that Φ is a local superdiffeomorphism at p. Then, there exist charts U and V around p and $\phi(p)$ respectively, such that $\phi(U) \subset V$ and $\phi|_U : U \to V$ is an isomorphism.*

PROOF. This proof relies heavily on the classical inverse function theorem and can be found in [CCF11, Proposition 5.1.1]. □

We end this section by briefly introducing immersions and submersions, noting that the differences to the non-super theory are only minor. A detailed treatment of this theory can be found in [CCF11, Lei80, Var04]. Given a supermanifold morphism $\Phi : \mathscr{M} \to \mathscr{N}$, we say Φ is an *immersion* (resp. *submersion*) at $p \in M$ if the differential $\mathrm{d}\Phi_p$ is injective (resp. surjective).

2.6. Lie supergroups

This section introduces the theory of Lie supergroups. Similarly to the usual definition of a Lie group, we can define a Lie supergroup as a group object in the category of supermanifolds. Alternatively, following Kostant [Kos77], we can define a Lie supergroup by its underlying real Lie group and an associated Lie superalgebra. We introduce both definitions and discuss their equivalence.

Throughout this section, we will make abundant use of the fact that a supermanifold morphism is determined by its image on global sections. In the following, note that the supermanifold $\mathbb{R}^{0,0}$ is simply a point equipped with the sheaf $\mathbb{R}$.

For any supermanifold $\mathscr{M}$, each point $p \in M$ defines an embedding $p = (\delta_p, \delta_p^*)$ of $\mathbb{R}^{0,0}$ into $\mathscr{M}$, where we define $\delta_p^* : \mathscr{O}_M(M) \to \mathbb{R}$ to be the evaluation map $f \mapsto \tilde{f}(p)$. Define the constant map $\hat{p} = (\hat{\delta}_p, \hat{\delta}_p^*) : \mathscr{M} \to \mathscr{M}$ as the composition of $p : \mathbb{R}^{0,0} \to \mathscr{M}$ with the unique map from $\mathscr{M}$ to $\mathbb{R}^{0,0}$.

A *Lie supergroup* G is a real supermanifold with smooth multiplication, inversion and unit maps

$$\mu = (m, m^*) : G \times G \to G, \quad \iota = (i, i^*) : G \to G, \quad e = (\delta_e, \delta_e^*) : \mathbb{R}^{0,0} \to G,$$

such that the following commute:

$$
\begin{array}{ccc}
G \times G \times G & \xrightarrow{\mu \times \mathrm{id}} & G \times G \\
\downarrow{\scriptstyle \mathrm{id} \times \mu} & & \downarrow{\scriptstyle \mu} \\
G \times G & \xrightarrow{\mu} & G
\end{array}
,\qquad
\begin{array}{ccccc}
 & & G \times G & & \\
 & \nearrow^{\langle \mathrm{id}_G, \hat{e} \rangle} & & \searrow^{\mu} & \\
G & & \xrightarrow{\quad \mathrm{id}_G \quad} & & G \\
 & \searrow_{\langle \hat{e}, \mathrm{id}_G \rangle} & & \nearrow_{\mu} & \\
 & & G \times G & &
\end{array}
,\qquad
\begin{array}{ccccc}
 & & G \times G & & \\
 & \nearrow^{\langle \mathrm{id}_G, \iota \rangle} & & \searrow^{\mu} & \\
G & & \xrightarrow{\quad \hat{e} \quad} & & G \\
 & \searrow_{\langle \iota, \mathrm{id}_G \rangle} & & \nearrow_{\mu} & \\
 & & G \times G & &
\end{array}
.
$$

Here $\langle \phi, \psi \rangle (\cdot) = \phi(\cdot) \times \psi(\cdot)$ denotes the diagonal map. These diagrams amount to the usual group laws of associativity, inverses and identity:

(i) $\mu \circ (\mathrm{id}_G \times \mu) = \mu \circ (\mu \times \mathrm{id}_G)$,
(ii) $\mu \circ \langle \mathrm{id}_G, \iota \rangle = \mu \circ \langle \iota, \mathrm{id}_G \rangle = \hat{e}$, and
(iii) $\mu \circ \langle \mathrm{id}_G, \hat{e} \rangle = \mu \circ \langle \hat{e}, \mathrm{id}_G \rangle$.

It is important to remark that each of these diagrams corresponds to two diagrams: one for the topological map, and one for the sheaf map. Considering only the topological maps, we see that the reduced manifold G_{Red} is a (non-super) Lie group.

Since morphisms of supermanifolds are determined by their image on global sections, we may equivalently state (i) – (iii) as follows:

(a) $(\mathrm{id} \otimes m^*) \circ m^* = (m^* \otimes \mathrm{id}) \circ m^*$ as a map $\mathscr{O}_G(G) \to \overline{\mathscr{O}_G(G) \otimes \mathscr{O}_G(G) \otimes \mathscr{O}_G(G)}$,
(b) $(\mathrm{id} \otimes \delta_e^*) \circ m^* = (\delta_e^* \otimes \mathrm{id}) \circ m^* = I$ as a map $\mathscr{O}_G(G) \to \mathscr{O}_G(G)$, and
(c) $m_{\mathscr{O}_G} \circ (\mathrm{id} \otimes i^*) \circ m^* = m_{\mathscr{O}_G} \circ (i^* \otimes \mathrm{id}) \circ m^* = \delta_e^*$ as a map $\mathscr{O}_G \to \mathbb{R}$.

REMARK 2.17. *A technicality arises when considering this point of view: it is not the case that $\mathscr{O}_{M \times M}(M \times M) = \mathscr{O}_M(M) \otimes \mathscr{O}_M(M)$. To get around this we take the completion of the tensor product, which we denote by $\overline{\mathscr{O}_M \otimes \mathscr{O}_M(M)}$.*

Intuitively, we can translate definitions from Lie theory to the super setting by using (a) – (c) in place of the traditional group laws. For instance, on a Lie group G, left-translation $L_g : G \to G$ is given by $h \mapsto gh$. More abstractly, we may view left-translation as a map $G \cong \{g\} \times G \hookrightarrow G \times G \xrightarrow{m} G$ by identifying G with $\{g\} \times G$ and mapping this into G via multiplication. This adapts to the super setting well:

Given a Lie supergroup G, for each $g \in G$ we define *left* and *right-translations by g* to be the supermanifold morphisms $L_g : G \to G$ and $R_g : G \to G$ given by $L_g^*(f) = (\delta_g^* \otimes I)(m^* f)$ and $R_g^*(f) := (I \otimes \delta_g^*)(m^* f)$, respectively. As in the non-super case, L_g and R_g are superdiffeomorphisms whose inverses are given by $(L_g)^{-1} = L_{g^{-1}}$ and $(R_g)^{-1} = R_{g^{-1}}$, respectively. Furthermore, we have the property that for any $g, h \in G$, $L_g \circ L_h = L_{gh}$ and $R_g \circ R_h = R_{hg}$ [BSV91, Propositions 2.2 and 2.3].

To extend the concept of the Lie algebra of a Lie group to the super case, we need to define left-invariant supervector fields. A (non-super) vector field is left-invariant if $(\mathrm{d}L_g)_h(X_h) = X_{gh}$ for all $g \in G$, or equivalently if $\mathrm{d}L_g X = X$. This is the definition in the super case: a supervector field $X \in \mathscr{T}_G(G)$ is *left-invariant* if $(I \otimes X) \circ m^* = m^* \circ X$.

The *Lie superalgebra* of a Lie supergroup G, denoted by $\mathfrak{g}$, is the set of all left-invariant vector fields on G. As in the non-super theory, $\mathfrak{g}$ is a finite-dimensional vector superspace that we identify with the tangent space at the identity of G. We give $\mathfrak{g}$ the structure of a Lie superalgebra by equipping it with the supercommutator

$$[X, Y] := XY - (-1)^{|X||Y|} YX.$$

Expectedly, the even component $\mathfrak{g}_0$ of $\mathfrak{g}$, is the Lie algebra of the underlying group G_{Red}. We now have two well understood objects associated with each Lie supergroup: the underlying Lie group, and the Lie superalgebra. It turns out that this is enough to determine the Lie supergroup completely.

A *super Harish-Chandra pair* (sHCp) is a pair $(G_{\mathrm{Red}}, \mathfrak{g})$ consisting of a Lie group and a Lie superalgebra such that

(i) $\mathfrak{g}_0 \cong \mathrm{Lie}(G_{\mathrm{Red}})$, and

(ii) there exists an action $\mathrm{Ad} : G_{\mathrm{Red}} \to \mathrm{Aut}(\mathfrak{g})$ such that $\widetilde{\mathrm{Ad}} : G_{\mathrm{Red}} \to \mathrm{Aut}(\mathfrak{g}_0)$ is the usual adjoint action. Furthermore, for all $x \in \mathfrak{g}_0$, and $Y \in \mathfrak{g}$,

$$(\mathrm{d}\,\mathrm{Ad})(X)Y = \mathrm{ad}_X Y = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Ad}_{\exp(tX)} Y = [X,Y].$$

The following result allows us to use either formulation of Lie supergroups without worry.

THEOREM 2.18. *Any Lie supergroup $G = (G_{\mathrm{Red}}, \mathscr{O}_G)$ defines a super Harish-Chandra pair $(G_{\mathrm{Red}}, \mathfrak{g})$ where the adjoint action* $\mathrm{Ad} : G_{\mathrm{Red}} \to \mathrm{Aut}(\mathfrak{g})$ *is given by* $\mathrm{Ad}_g(X) = R_g^* \circ X \circ R_{g^{-1}}^*$. *There exists a one-to-one correspondence*

$$\left\{\begin{matrix}\textit{Group objects in the category of}\\ \textit{supermanifolds } (G_{Red}, \mathscr{O}_G)\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\textit{Super Harish-Chandra pairs}\\ (G_{Red}, \mathfrak{g})\end{matrix}\right\}.$$

PROOF. For a complete proof of this result, see [CCF11, Theorem 7.4.5]. □

EXAMPLE 2.19. *We will now give some important examples of Lie supergroups in terms of their super Harish-Chandra pairs:*

(i) The general linear supergroup $\mathsf{GL}(m|n)$ *is the Lie supergroup associated to the sHCp*

$$(\mathsf{GL}(m) \times \mathsf{GL}(n), \mathfrak{gl}(m|n)).$$

(ii) The special linear supergroup $\mathsf{SL}(m|n)$ *is the subgroup of* $\mathsf{GL}(m|n)$ *with reduced group*

$$\mathsf{SL}(m|n)_{\mathrm{Red}} := \{(A,B) \in \mathsf{GL}(m) \times \mathsf{GL}(n) \mid \det A = \det B > 0\} \cong \mathsf{SL}(m) \times \mathsf{SL}(n) \times \mathbb{R},$$

and Lie superalgebra

$$\mathfrak{sl}(m|n) := \{X \in \mathfrak{gl}(m|n) \mid \mathrm{str} X = 0\}.$$

(iii) The orthosymplectic supergroup *Let $V = V_0 \oplus V_1$ be a complex vector superspace and $\omega : V \times V \to \mathbb{C}$ a non-degenerate supersymmetric even bilinear form. Assume $\dim V_0 = m$ and $\dim V_1 = 2n$ for positive integers m and n. The orthosymplectic superalgebra is defined to be*

$$\mathfrak{osp}(m|2n)^{\mathbb{C}} := \left\{X \in \mathfrak{gl}(m|2n)^{\mathbb{C}} : \omega(Xu,v) + (-1)^{|X||u|}\omega(u,Xv) = 0 \textit{ for all } u,v \in V\right\}.$$

The compact real form of $\mathfrak{osp}(m|2n)^{\mathbb{C}}$ is defined by

$$\mathfrak{osp}(m|2n)_0 = \mathfrak{so}(2,\mathbb{R}) \oplus \mathfrak{sp}(n) \cong \mathfrak{so}(2,\mathbb{R}) \oplus \mathfrak{u}(n,\mathbb{H}), \textit{ and}$$
$$\mathfrak{osp}(m|2n)_1 \cong \mathbb{H}^n.$$

Denote by $\mathsf{SOSp}(2|2n)$ *the connected Lie supergroup with compact real form* $\mathfrak{osp}(2|2n)$ [GPRZ23, Section 7.1].

(iv) *The special unitary supergroup* $\mathsf{SU}(m|n)$ *is the Lie supergroup with associated sHCp*

$$(\mathsf{SU}(m)\times\mathsf{SU}(n),\mathfrak{su}(m|n)),$$

where

$$\mathfrak{su}(m|n) := \left\{ X = \begin{pmatrix} A & B \\ -iB^* & C \end{pmatrix} \middle| A^* = -A, C^* = -C, \operatorname{str} X = 0 \right\}.$$

2.7. Lie supergroup actions

In this section, we discuss Lie supergroup actions on supermanifolds. This gives us the required language to discuss homogeneous supermanifolds.

Let $G = (G_{\mathrm{Red}}, \mathscr{O}_G)$ be a Lie supergroup and $\mathscr{M} = (M, \mathscr{O}_M)$ be a supermanifold. G *acts* on $\mathscr{M}$ if there exists a morphism $\Psi = (\psi, \psi^*) : G \times \mathscr{M} \to \mathscr{M}$ such that

(i) $\Psi \circ (\mathrm{id}_G \times \Psi) = \Psi \circ (\mu \times \mathrm{id}_M)$ as maps $G \times G \times \mathscr{M} \to \mathscr{M}$, and
(ii) $\Psi \circ (\hat{e} \times \mathrm{id}_M) = \mathrm{id}_M$ as maps $\mathscr{M} \to \mathscr{M}$.

These conditions correspond to the familiar axioms: $g \cdot (h \cdot x) = gh \cdot x$, and $e \cdot x = x$. The triple $(M, \mathscr{O}_M, \Psi)$ is referred to as a *G-supermanifold.*

Alternatively, in the language of super Harish-Chandra pairs, an *action* of $(G_{\mathrm{Red}}, \mathfrak{g})$ on $\mathscr{M}$ is a pair $(\rho, \hat{\rho})$ consisting of a group homomorphism $\rho : G_{\mathrm{Red}} \times \mathscr{M} \to \mathscr{M}$ and a Lie supergroup anti-homomorphism $\hat{\rho} : \mathfrak{g} \to \mathscr{T}_M(M)$ such that

$$\hat{\rho}(X)(f) = (\mathrm{d}\rho)(X)(f) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\rho \circ \exp(tX))^*(f)$$

for all $f \in \mathscr{O}_M(M)$ and all $X \in \mathfrak{g}$.

If a (non-super) Lie group G acts on a smooth manifold M, then for any $X \in \mathrm{Lie}(G)$ and $p \in M$, we can define a vector field $\hat{X} \in \mathfrak{X}(M)$ to be the infinitesimal generator of the induced flow of X

$$\hat{X}_p = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (p\exp(tX)).$$

The vector field $\hat{X}$ is known as the *action field of* X. This looks very similar to our $\hat{\rho}$ map. In fact, the map taking a vector field to its action field is an anti-homomorphism too! We draw attention to the fact that for each $g \in G_{\mathrm{Red}}$, $\rho(g) : \mathscr{M} \to \mathscr{M}$ is a supermanifold morphism. Hence, to define an action $(\rho, \hat{\rho})$ of a sHCp on a supermanifold, we require two pieces of data:

(i) the image of global sections of $\mathscr{O}_M$ under $(\rho(g))^*$ for all $g \in G_{\mathrm{Red}}$, and
(ii) the image of global sections of $\mathscr{T}_M$ under $\hat{\rho}$.

The two given definitions of a supergroup action are equivalent. Any action $\Psi : G \times \mathscr{M} \to \mathscr{M}$ gives rise to an action $(\rho, \hat{\rho})$ of the associated sHCp $(G_{\mathrm{Red}}, \mathfrak{g})$ on $\mathscr{M}$ with the following two maps:

(i) $(\rho(g))^* = (\delta_g \otimes \mathrm{id}) \circ \psi^*$ for all $g \in G_{\mathrm{Red}}$, and
(ii) $\hat{\rho}(X) = (X|_e \otimes \mathrm{id}) \circ \psi^*$ for all $X \in \mathfrak{g}$.

For the proof of the converse assertion, see [CCF11, Proposition 8.3.2].

EXAMPLE 2.20. *The adjoint action* Ad^G *of a Lie supergroup* $G = (G_{\mathrm{Red}}, \mathscr{O}_G)$ *on itself is defined by the following action of the associated sHCp* $(G_{\mathrm{Red}}, \mathfrak{g})$ *on G:*

(i) $(\rho(g))^* = L_g^* \circ R_{g^{-1}}^* =: \mathrm{Ad}_g^*$, *and*
(ii) $\hat{\rho}(X) = (X|_e \otimes \mathrm{id}) \circ m^* - X$

for all $g \in G_{\mathrm{Red}}$, $X \in \mathfrak{g}$.

2.8. Some basic representation theory

Here we present some basic representation theory, focusing on (non-super) Lie groups. Throughout this section, let G be a Lie group.

A *representation* of a Lie group G on a vector space V over a field k is a homomorphism $\rho : G \to \mathsf{GL}(V)$ given by $g \mapsto \rho(g)$ such that $\rho(g_1 g_2) = \rho(g_1)\rho(g_2)$ for all $g_1, g_2 \in G$. The *dimension* of a representation ρ is the dimension of the vector space V.

A *G-module* is a vector space V equipped with an operation $G \times V \to V$ given by $(g, v) \mapsto gv$ satisfying

(i) $(ag + bh)v = a(gv) + b(hv)$, and
(ii) $g(av + bw) = a(gv) + b(gw)$

for all $g, h \in G$, $a, b \in k$ and $v, w, \in V$.

Every representation ρ of G on V, defines a G-module. Indeed, for $g \in G$, $\rho(g) \in \mathsf{GL}(V)$ acts on vectors via $\rho(g)(v) = gv$. In fact, the converse holds too: an abstract G-module V defines a group action, which in turn defines a representation.

Let V and W be vector spaces with representations $\rho_1 : G \to \mathsf{GL}(V)$ and $\rho_2 : G \to \mathsf{GL}(W)$. We say that a map $f : V \to W$ *intertwines* ρ_1 and ρ_2 if $f \circ \rho_1 = \rho_2 \circ f$. We say that the two representations or G-modules are *equivalent* if there exists an intertwiner $f : V \to W$.

Let $\rho : G \to \mathsf{GL}(V)$ be a representation of G on V. Given a subspace $U \subset V$, we say that U is *G-invariant* if $gU \subset U$, for all $g \in G$. Every representation $\rho : G \to \mathsf{GL}(V)$ has at least two invariant subspaces: $\{0\}$, and V itself. We say that a representation is *irreducible* if the only invariant subspaces are these two.

THEOREM 2.21 (Schur's Lemma – Part 1). *Let V and W be two irreducible representations of G, and let $f : V \to W$ be an intertwiner. Then either f is bijective or $f = 0$.*

PROOF. It is easy to see that $\ker f = \{v \in V : f(v) = 0_W\} \subset V$ and $\operatorname{im} f = \{f(v) : v \in V\} \subset W$ are G-invariant. As the representations are irreducible, $\ker f$ is either $\{0\}$ or V, and $\operatorname{im} f$ is either $\{0\}$ or W. If $\ker f = V$, then $\operatorname{im} f = \{0\}$ and so $f = 0$. On the other hand, if $\ker f = \{0\}$ then $\operatorname{im} f = W$ and f is bijective. □

THEOREM 2.22 (Schur's Lemma – Part 2). *If V is an irreducible representation over $\mathbb{C}$ and $f : V \to V$ is an intertwiner, then $f = \lambda \mathrm{Id}_V$.*

PROOF. By the fundamental theorem of algebra, f has an eigenvalue $\lambda \in \mathbb{C}$. Notice then that $f - \lambda \mathrm{Id}_V$ intertwines V, which is irreducible. Also, $f - \lambda \mathrm{Id}_V$ is not bijective, and so by the above theorem, must be the zero map. Hence, $f = \lambda \mathrm{Id}_V$. □

Irreducible representations are the building blocks of all other representations. One can decompose any representation into its irreducible components. To see this, we need the existence of an inner product.

THEOREM 2.23. [Arv03, Theorem 2.5] *Let $\rho : G \to \mathsf{GL}(V)$ be a representation of a compact group G. Then there exists a G-invariant inner product $\langle \cdot, \cdot \rangle$ on V.*

THEOREM 2.24. [Arv03, Theorem 2.6] *Any finite-dimensional representation of a compact group is a direct sum of irreducible representations.*

PROOF. Let G be a compact Lie group and V a G-module. If V is irreducible we are done. Assume V is not irreducible and let $\langle \cdot, \cdot \rangle$ be a G-invariant inner product. As V is not irreducible, there exists a proper subspace $U \subset V$ that is a non-trivial submodule. Consider $U^{\perp}$; for any $u \in U$ and $w \in U^{\perp}$,

$$\langle gu, gw \rangle = \langle u, w \rangle = 0.$$

We see $U^{\perp}$ is G-invariant since

$$\langle gu, w \rangle = \left\langle u, g^{-1}w \right\rangle = 0$$

for all $g \in G, u \in U, w \in U^{\perp}$. Also, $V = U \oplus U^{\perp}$. Repeat this process inductively to obtain the result. □

The decomposition obtained above is not always unique. We say that the summands V_i are *monotypic* if they are pairwise inequivalent. It holds then that a monotypic decomposition is equivalent up to the ordering of the summands.

CHAPTER 3

Riemannian supergeometry

The field of supergeometry revolves around the measurement of distances, angles, and volumes on abstract supermanifolds. To address this, we introduce the concept of a Riemannian metric in the setting of supermanifolds.

Since the choice of metric determines the manifold's geometry, it is vital to understand when two metrics give the 'same' geometry. This equivalence is known as isometry. We discuss geometric properties that are preserved under isometry, one of the most pronounced of which is the curvature. We conclude the chapter by discussing homogeneous superspaces.

3.1. Graded Riemannian metrics

Let $\mathscr{M} = (M, \mathscr{O}_M)$ be an (m,n)-dimensional supermanifold. A *graded Riemannian metric* is a non-degenerate graded-symmetric even $\mathscr{O}_M$-linear morphism of sheaves $g(\cdot,\cdot) : \mathscr{T}_M \otimes \mathscr{T}_M \to \mathscr{O}_M$. In this setting, the non-degeneracy means that the map $X \mapsto g(X,\cdot)$ is an isomorphism. If $\mathscr{M}$ has a Riemannian metric g, we call $(\mathscr{M}, g)$ a *Riemannian supermanifold*.

For any point $p \in M$, $g(\cdot,\cdot)$ establishes a scalar superproduct $\langle \cdot,\cdot \rangle_p$ on the tangent space T_pM. However, unlike the non-super theory, knowledge of the scalar product at every point does not determine the metric on $\mathscr{M}$ in general. Despite this, we often write $\langle \cdot,\cdot \rangle$ when we mean $g(\cdot,\cdot)$.

It is apparent that the symmetric scalar product $\langle \cdot,\cdot \rangle_{p,0}$ on T_pM_{Red} gives rise to a pseudo-Riemannian metric on M_{Red}. In contrast to Riemannian metrics, not all supermanifolds possess a graded Riemannian metric. In fact, due to the symplectic nature of $\langle \cdot,\cdot \rangle_1$, only supermanifolds with an even-dimensional odd component admit a graded Riemannian metric.

3.2. Vector bundles and connections

We now introduce the concept of vector bundles and locally free sheaves. This serves as a foundation for establishing the definition of a connection on the tangent sheaf of a supermanifold.

Let M be a topological space. A *real vector bundle* of rank k over M is a topological space E and a surjective map $\pi : E \to M$ such that two conditions hold:

(i) for every point $p \in M$, the fibre $\pi^{-1}(p) =: E_p$ is endowed with the structure of a k-dimensional vector space;

(ii) for every point $p \in M$, there exists a neighbourhood $U \ni p$ and a homeomorphism $\phi : U \times \mathbb{R}^k \to \pi^{-1}(U)$ (called a local trivialisation) such that $(\pi \circ \phi)(p,v) = p$ for all $v \in \mathbb{R}^k$, and the map $v \mapsto \phi(p,v)$ is an isomorphism of $\mathbb{R}^k$ and $\pi^{-1}(p)$.

We refer to E as the total space, M as the base space, and π as the projection. We say that a *section* of $\pi : E \to M$ is a continuous map $\sigma : M \to E$ such that $\pi \circ \sigma = \text{Id}_M$.

EXAMPLE 3.1. *The tangent bundle TM of a smooth manifold M is a vector bundle with the standard projection map. The fibres are the tangent spaces of M, endowed with their natural vector space structure. The sections of the projection map are the smooth vector fields on M. This language should feel familiar from our discussion of sheaves in Section 2.3.*

Let $(X,\mathscr{F})$ be a ringed space. A sheaf of $\mathscr{F}$-modules $\mathscr{G}$ is said to be *free* if $\mathscr{G} = \bigoplus_{i=1}^{n}\mathscr{F}$. We say that $\mathscr{G}$ is *locally free* if there exists an open cover $\{U_\alpha\}_{\alpha\in I}$ of X such that the restriction $\mathscr{G}|_{U_\alpha}$ is a free sheaf of $\mathscr{F}|_U$-modules. If X is connected and n is constant over every point in X, then we say $\mathscr{G}$ is a locally free sheaf of *rank n*.

EXAMPLE 3.2. *Consider a vector bundle (E,π,M) of rank n over M. Take a point $p\in M$ and an open neighbourhood $U\subset M$ containing p. Let $\Gamma(E,U)$ denote the space of local sections, which are continuous maps $\sigma : U\to E$ such that $\pi|_U\circ\sigma = \mathrm{Id}_U$. This becomes a vector space when equipped with scalar multiplication and pointwise addition. The mapping $U\to\Gamma(E,U)$ establishes a presheaf with regular function restriction, and, through sheafification, forms a sheaf $\mathscr{E}$. Let $\mathscr{O}_M$ be the sheaf of continuous functions on M. It turns out that $\mathscr{E}$ is a locally free sheaf of $\mathscr{O}_M$-modules.*

To illustrate this, let $\sigma\in\mathscr{E}(U)$ and $f\in\mathscr{O}_M(U)$. Suppose $\sigma(p) = (p,v)\in\{p\}\times\mathbb{R}^n$ and define the multiplication $f\sigma(p) = (p, f(p)v)\in\{p\}\times\mathbb{R}^n$. Due to the local trivialisation $\pi^{-1}(p)\cong U\times\mathbb{R}^n$, it is evident that $f\sigma\in\mathscr{E}(U)$, rendering $\mathscr{E}(U)$ an $\mathscr{O}_M(U)$-module. Furthermore, a continuous section $U\to U\times\mathbb{R}^n$ can be expressed as n continuous maps from U to $\mathbb{R}$. This shows that

$$\mathscr{E}(U)\cong\bigoplus_{i=1}^{n}\mathscr{O}_M(U),$$

demonstrating $\mathscr{E}$'s local freeness.

Conversely, it is possible to construct a distinct vector bundle of rank n from a locally free sheaf of $\mathscr{O}_M$-modules with rank n. This correspondence motivates the definition of a supervector bundle, and consequently the super tangent bundle. We say that a *supervector bundle* over $\mathscr{M}$ is a rank (p,q) locally free sheaf of $\mathscr{O}_M$-modules on $\mathscr{M}$.

EXAMPLE 3.3. *Both the tangent sheaf $\mathscr{T}_M$ and the sheaf of vector fields along a morphism $\mathscr{T}_\Phi$ are supervector bundles. Equations (2.3) and (2.4) demonstrate their fulfilment of the locally free property.*

Another important vector bundle on $\mathscr{M}$ is the *cotangent bundle* Ω^1_M defined as the dual of the tangent bundle. This duality is described by the map

$$\langle\cdot,\cdot\rangle : \mathscr{T}_M\otimes_{\mathscr{O}_M}\Omega^1_M\to\mathscr{O}_M,$$

where $\langle uX, v\omega\rangle = (-1)^{|X||v|}uv\langle X,\omega\rangle$ for $u,v\in\mathscr{O}_M$. We define the *differential* of a function to be the map $\mathrm{d} : \mathscr{O}_M\to\Omega^1_M$ where $\langle X,\mathrm{d}f\rangle = Xf$.

In geometry, the concept of a connection is crucial as it allows us to make sense of parallel transport – a way of connecting fibres of a vector bundle over nearby points.

A *connection* on a supervector bundle $\mathscr{E}$ over a supermanifold $\mathscr{M}$ is defined to be an even morphism of sheaves

$$\nabla : \mathscr{E}\to\Omega^1_M\otimes\mathscr{E}$$

such that for all $f\in\mathscr{O}_M(M)$, and $v\in\mathscr{E}$, $\nabla(fv) = \mathrm{d}f\otimes v + f\nabla v$. Then, by defining $\langle X,\alpha\otimes v\rangle := \langle X,\alpha\rangle v$ for $\alpha\in\Omega^1_M$, we can use the connection to differentiate fv in the direction of X:

$$\nabla_X fv = Xf(v) + (-1)^{|X||f|}f\nabla_X v, \tag{3.1}$$

where the parity of $\nabla_X v$ is $|X|+|v|$. Consider a connection ∇ on the tangent bundle $\mathscr{T}_M$. We define the *torsion* of ∇ by

$$T_\nabla(X,Y) := \nabla_X Y - (-1)^{|X||Y|}\nabla_Y X - [X,Y].$$

This quantity measures the failure of the connection to be commutative. In some sense, it tells us how tangent spaces twist along a curve during parallel transport.

In the context of a Riemannian supermanifold, we desire a connection on $\mathscr{T}_M$ that interacts well with the metric. If $\mathscr{M}$ is endowed with a graded Riemannian metric g, we say a connection ∇ *metric-compatible* if it satisfies

$$X\langle Y,Z\rangle = \langle \nabla_X Y, Z\rangle + (-1)^{|X||Y|}\langle Y, \nabla_X Z\rangle.$$

THEOREM 3.4 (Fundamental Theorem of Riemannian Geometry). *On a Riemannian supermanifold $(\mathscr{M},g)$, there exists a unique torsionless, metric compatible connection ∇, which is implicitly defined by*

$$\begin{aligned}2\langle \nabla_X Y, Z\rangle &= X\langle Y,Z\rangle - (-1)^{|Z|(|X|+|Y|)}Z\langle X,Y\rangle + (-1)^{|X|(|Y|+|Z|)}Y\langle Z,X\rangle \\ &\quad + \langle [X,Y],Z\rangle - (-1)^{|X|(|Y|+|Z|)}\langle [Y,Z],X\rangle + (-1)^{|Z|(|X|+|Y|)}\langle [Z,X],Y\rangle.\end{aligned} \tag{3.2}$$

We call ∇ the Levi-Civita connection on $\mathscr{M}$.

PROOF. Apart from the extra signs, the proof is exactly as in [Lee18, Theorem 5.10]. □

If we consider purely even vector fields in (3.2), we arrive at Koszul's formula, thus defining the standard Levi-Civita connection on (M_{Red}, g_0). Here, g_0 is the pseudo-Riemannian metric obtained by restricting g to the even tangent spaces.

3.3. Curvature and isometries

In geometry, we often ask if two objects are the same or not. One approach to answering this question is to look for local invariants. These are quantities that remain the same under geometric transformations, such as a rotation or translation. For instance, the number of angles and sides of a polygon are local invariants.

Within Riemannian geometry, local invariants play a pivotal role in demonstrating that two manifodls aren't locally isometric – one cannot be transformed into the other without altering its underlying structure. In this section, we explore a fundamental invariant: curvature.

3.3.1. Curvature. Fix ∇ to be the Levi-Civita connection on a supermanifold $\mathscr{M}$. The *curvature* of ∇ is defined to be the map $R : \mathscr{T}_M(M)\times\mathscr{T}_M(M)\times\mathscr{T}_M(M)\to\mathscr{T}_M(M)$ given by

$$R(X,Y)Z := [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z. \tag{3.3}$$

We often find that the information encoded in the Riemannian curvature is more conveniently represented using the *Riemann curvature tensor*, a $(0,4)$-tensor field defined by $\mathrm{Rm}(X,Y,Z,W) := \langle R(X,Y)Z, W\rangle$. The curvature tensor possesses several symmetries.

PROPOSITION 3.5 (Symmetries of the Curvature Tensor). *Let $(\mathscr{M},g)$ be a Riemannian supermanifold equipped with the Levi-Civita connection. The Riemann curvature tensor* Rm *has the following symmetries for all vector fields $X,Y,Z,W \in \mathscr{T}_M(M)$:*

(i) $-\mathrm{Rm}(X,Y,Z,W) = (-1)^{|X||Y|}\mathrm{Rm}(Y,X,Z,W)$,
(ii) $-\mathrm{Rm}(X,Y,Z,W) = (-1)^{|Z||W|}\mathrm{Rm}(Y,X,W,Z)$,
(iii) $\mathrm{Rm}(X,Y,Z,W) = (-1)^{(|X|+|Y|)(|Z|+|W|)}\mathrm{Rm}(Z,W,X,Y)$, *and*

(iv) $\mathrm{Rm}(X,Y,Z,W) + (-1)^{|Z|(|X|+|Y|)}\mathrm{Rm}(Z,X,Y,W) + (-1)^{|X|(|Y|+|Z|)}\mathrm{Rm}(Y,Z,X,W) = 0.$

PROOF. We only prove (i) to demonstrate how the argument in [Lee18, Proposition 7.12] adapts to the extra signs. Indeed,

$$\begin{aligned}
-\mathrm{Rm}(X,Y,Z,W) &= -\langle R(X,Y)Z, W\rangle \\
&= -\left\langle \nabla_X\nabla_Y Z - (-1)^{|X||Y|}\nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, W\right\rangle \\
&= -\left\langle \nabla_X\nabla_Y Z - (-1)^{|X||Y|}\nabla_Y\nabla_X Z + (-1)^{|X||Y|}\nabla_{[Y,Z]}Z, W\right\rangle \\
&= (-1)^{|X||Y|}\langle R(Y,X)Z, W\rangle \\
&= (-1)^{|X||Y|}\mathrm{Rm}(Y,X,Z,W).
\end{aligned}$$

□

Computing the Riemann curvature tensor is often a challenging task due to its wealth of information. This complexity prompts us to explore alternative constructions that capture the tensor's essence while being simpler to compute. One such tensor that has been studied extensively is the $(0,2)$-tensor known as the *Ricci tensor* and denoted by Ric. For vector fields X and Y, we define

$$\mathrm{Ric}(X,Y) = \mathrm{str}(Z \mapsto (-1)^{|Z||Y|}R(X,Z)Y) \tag{3.4}$$

where str is the supertrace from Section 2.1. The Ricci curvature exhibits graded symmetry: for all vector fields X,Y, $\mathrm{Ric}(X,Y) = (-1)^{|X||Y|}\,\mathrm{Ric}(Y,X)$. Taking the supertrace of the Ricci tensor, we obtain the *scalar curvature* S.

We often also consider the *Ricci endomorphism*, a $(1,1)$-tensor denoted by $\mathrm{ric}: \mathscr{T}_M \to \mathscr{T}_M$ and defined by

$$\mathrm{Ric}(g)(\cdot,\cdot) = \langle \mathrm{ric}(\cdot), \cdot\rangle .$$

3.3.2. Isometries and the isometry group. Let $(\mathscr{M}, g_1)$ and $(\mathscr{N}, g_2)$ be Riemannian supermanifolds. We say that a diffeomorphism $\Phi : \mathscr{M} \to \mathscr{N}$ is an *isometry* if it respects the metric. In other words, for vector fields $X,Y \in \mathscr{T}_M(M)$,

$$\phi^* \langle \mathrm{d}\Phi(X), \mathrm{d}\Phi(Y)\rangle_{g_2} = \phi^* \left\langle (\phi^{-1})^* \circ X \circ \phi^*, (\phi^{-1})^* \circ Y \circ \phi^*\right\rangle_{g_2} = \langle X, Y\rangle_{g_1}.$$

If $\Phi : \mathscr{M} \to \mathscr{N}$ is an isometry, we find

$$\mathrm{d}\Phi(\nabla_X Y) = \nabla_{\mathrm{d}\Phi X}\mathrm{d}\Phi Y$$

for vector fields $\mathrm{d}\Phi X, \mathrm{d}\Phi Y$, and $\mathrm{d}\Phi(\nabla_X Y)$ on $\mathscr{N}$. Although vector fields and scalar products aren't determined by their local values as in the non-super theory, the following remains true in both settings.

PROPOSITION 3.6. [Goe08, Proposition 6] *An isometry of a connected Riemannian supermanifold $\mathscr{M}$ is determined by its value and its derivative at one point.*

The set of isometries $I(M)$ of a Riemannian supermanifold $\mathscr{M}$ forms a supergroup which we call the *isometry group*. The Myers-Steenrod theorem establishes that the isometry group of a (non-super) Riemannian manifold possesses the structure of a Lie group. This remains the case for supermanifolds. To demonstrate this, we construct a super Harish-Chandra pair whose reduced group corresponds to the manifold's group of isometries.

Denote by $I(M)_{\text{Red}}$ the group of isometries of $\mathscr{M}$ (distinct from $I(M_{\text{Red}})$, the isometry group of the reduced manifold M_{Red}). It can be shown that $I(M)_{\text{Red}}$ has the structure of a Lie group [Goe08]. Recall that for a super Harish-Chandra pair $(G,\mathfrak{g})$, we require $\text{Lie}(G_{\text{Red}}) = \mathfrak{g}_0$.

In the non-super theory, the Lie algebra of the isometry group of a manifold consists in a class of distinguished vector fields. We now define their super analogue: a *graded Killing vector field* on $\mathscr{M}$ is a vector field X such that

$$X\langle Y,Z\rangle = \langle [X,Y],Z\rangle + (-1)^{|X||Y|}\langle Y,[X,Z]\rangle$$

for all vector fields Y and Z. The vector space of all graded Killing fields forms a Lie superalgebra when equipped with the negative of the bracket induced by the Lie superalgebra of all vector fields.

It turns out that the left-invariant vector fields of $I(M)_{\text{Red}}$ (the elements of its Lie algebra) are exactly the even Killing vector fields [Goe08]. Denote by $\mathfrak{g}$ the Lie superalgebra of graded Killing vector fields on $\mathscr{M}$. For each $X \in \mathfrak{g}$ and an isometry $\varphi \in I(M)_{\text{Red}}$, there is an action

$$\text{Ad}_\varphi X := \text{d}\varphi(X) = (\varphi^{-1})^* \circ X \circ \varphi^*$$

which, on $\mathfrak{g}_0$, coincides with the action of $I(M)_{\text{Red}}$ on its Lie algebra. Thus, $(I(M)_{\text{Red}},\mathfrak{g})$ forms the super Harish-Chandra pair associated with the isometry group of $\mathscr{M}$.

In the classical theory, a pseudo-Riemannian metric $\langle\cdot,\cdot\rangle$ on a Lie group G is considered *left-invariant* if the left-translations are isometries. In other words,

$$\langle (dL_g)X, (\text{d}L_g)Y\rangle = \langle X,Y\rangle$$

holds for all $X,Y \in \mathfrak{g} = \text{Lie}(G)$ and all $g \in G$. Equivalently, a metric is left-invariant if for all $X,Y \in \mathfrak{g}$, $\langle X,Y\rangle$ is constant on G. We adopt this as the definition of a left-invariant metric on a Lie supergroup. We can similarly define *right-invariant* and *bi-invariant* metrics. Left-invariant metrics are distinguished as they correspond to scalar superproducts on the Lie algebra. Indeed, for $X,Y \in \mathfrak{g}$, since $\langle X,Y\rangle$ is a constant function on G, we can linearly extend the metric with respect to superfunctions on G.

A scalar superproduct $\langle\cdot,\cdot\rangle$ is $\text{Ad}^{G_{\text{Red}}}$-*invariant* if

$$\langle \text{Ad}_g X, \text{Ad}_g Y\rangle = \langle X,Y\rangle$$

for all $X,Y \in \mathfrak{g}$ and all $g \in G_{\text{Red}}$. It is $\text{ad}_{\mathfrak{g}}$-*invariant* if

$$\langle [X,Y],Z\rangle + (-1)^{|X||Y|}\langle Y,[X,Z]\rangle = 0$$

for all $X,Y,Z \in \mathfrak{g}$. We call a scalar superproduct Ad^G-*invariant* if it is both $\text{Ad}^{G_{\text{Red}}}$-invariant and $\text{ad}_{\mathfrak{g}}$-invariant.

THEOREM 3.7. [Goe08, Theorem 2] *Let G be a Lie supergroup with a left-invariant graded Riemannian metric $\langle\cdot,\cdot\rangle$. The metric is bi-invariant if and only if $\langle\cdot,\cdot\rangle$ is Ad^G-invariant.*

In the classical theory, it is known that for any representation of a compact Lie group G, there exists a G-invariant inner product on V. A G-invariant inner product of particular interest is the one induced by the adjoint representation, known as the Killing form of G.

Let $\mathfrak{g}$ be a finite-dimensional Lie superalgebra. Recall that each element $X \in \mathfrak{g}$ defines the adjoint endomorphism $\text{ad}_X : \mathfrak{g} \to \mathfrak{g}$. The *Killing form* of $\mathfrak{g}$ is the map $B : \mathfrak{g}\times\mathfrak{g} \to \mathbb{R}$

$$B(X,Y) := \text{str}(\text{ad}_X \circ \text{ad}_Y),$$

for all $X,Y \in \mathfrak{g}$. The Killing form is an even, graded-symmetric, $\text{ad}_{\mathfrak{g}}$-invariant bilinear form. In contrast to the non-super setting, B is not always non-degenerate, as we now see.

EXAMPLE 3.8. *The following examples can be found in* [Goe08]*:*

(i) For the Lie superalgebra $\mathfrak{sl}(m|n)$, the killing form is given by

$$B(X,Y) = 2(n-m)\,\mathrm{str}(XY).$$

It is non-degenerate unless $n = m$.

(ii) For the Lie superalgebra $\mathfrak{osp}(m|2n)$, the Killing form is

$$B(X,Y) = (m-2n-2)\,\mathrm{str}(XY).$$

It is non-degenerate unless $m = 2n+2$.

3.4. Homogeneous superspaces

In this section, we introduce the concept of homogeneous supermanifolds. In the non-super setting, a homogeneous space, loosely, 'looks the same' everywhere; any two points are related by some transformation. This property significantly simplifies the analysis and manipulation of these spaces, making them particularly 'nice' to study.

Let G be a (non-super) Lie group. We say that a smooth manifold M is homogeneous if G acts transitively on M. Equivalently, we can consider a G-homogeneous space M to be the quotient space G/H, where H is the isotropy group of G at the identity. For a comprehensive introduction to the geometry of homogeneous spaces, we highly recommend the outstanding book by Arvanitoyeorgos [Arv03].

Let $G = (G_{\mathrm{Red}}, \mathscr{O}_G)$ be a Lie supergroup and consider $(H, \mathfrak{h})$ a closed Lie sub-supergroup of G. Denote the canonical projection maps by

$$\pi : G_{\mathrm{Red}} \to G_{\mathrm{Red}}/H_{\mathrm{Red}}, \quad \mathrm{pr} : G \times K \to G.$$

Additionally, let $R_H : G \times H \to G$ be the right action of H on G, where H acts by multiplication. We understand that $G_{\mathrm{Red}}/H_{\mathrm{Red}}$ forms a smooth manifold. We will shortly see that an analogous result holds for G/H in the super setting.

For each open set $U \subset G_{\mathrm{Red}}/H_{\mathrm{Red}}$, define the subalgebra of R_H-invariant superfunctions

$$\mathscr{O}_{G/H}(U) := \{f \in \mathscr{O}_G(\pi^{-1}(U)) : R_H^* f = \mathrm{pr}^* f\} \subset \mathscr{O}_G(\pi^{-1}(U)).$$

Let $\mathscr{O}_{G/H}$ denote the sheaf defined by the mapping $U \mapsto \mathscr{O}_{G/H}(U)$. Then, $G/H := (G_{\mathrm{Red}}/H_{\mathrm{Red}}, \mathscr{O}_{G/H})$ is a supermanifold [Kos77, Theorem 3.9]. The inclusion map establishes a morphism of sheaves which subsequently defines the morphism of supermanifolds $G \to G/H$. Consider the map $T_eG \to T_oG/H$ given by $X \mapsto XH$. This is a surjective morphism of supervector spaces with kernel $\mathfrak{h}$. Accordingly, we make the identification

$$T_oG/H \cong \mathfrak{g}/\mathfrak{h},$$

where $o := H$ represents the identity coset. Let $\Psi = (\psi, \psi^*)$ be the action of a Lie supergroup G on a supermanifold $\mathscr{M}$. Each point $p \in M$ defines a closed Lie sub-supergroup G_p called the *isotropy group* of G at p. The isotropy group is defined by the sHCp $((G_{\mathrm{Red}})_p, \mathfrak{g}_p)$, where $(G_{\mathrm{Red}})_p := \{h \in G_{\mathrm{Red}} : h(p) = p\}$ is the usual isotropy group and $\mathfrak{g}_p := \{X \in \mathfrak{g} : \hat{X}|_p = 0\}$. In other words, $\mathfrak{g}_p$ is the left-invariant vector fields on G whose infinitesimal action at p is trivial. The adjoint action $\mathrm{Ad}^{G_{\mathrm{Red}}} : (G_{\mathrm{Red}})_p \to \mathsf{GL}(\mathfrak{g}_p)$ is the restriction of the adjoint action $\mathrm{Ad} : G_{\mathrm{Red}} \to \mathsf{GL}(\mathfrak{g})$.

The action of G_p on T_pM defines a linear representation $\mathrm{Ad}^{G/G_p} : G_p \to \mathsf{GL}(T_pM)$ of the isotropy group at p on the tangent space T_pM. This representation is known as the *isotropy representation* of M. The map Ad^{G/G_p} is defined by two maps:

(i) $(G_{\mathrm{Red}})_p \to \mathsf{GL}(T_pM)$, given by $h \mapsto (\mathrm{d}\psi(h))_p$, and

(ii) $\mathfrak{g}_p \to \mathfrak{gl}(T_pM)$, given by $X \cdot v = -[\hat{X}, v]|_p = (-1)^{|X||v|} v \circ \hat{X}$.

For each point $p \in M$, we define a submanifold $G \cdot p$ of $\mathscr{M}$ referred to as the *orbit* of G at p. The orbit $G_{\text{Red}} \cdot p = \{g \cdot p : g \in G_{\text{Red}}\}$ obtains the structure of a smooth manifold via the canonical mapping $j : G_{\text{Red}}/(G_{\text{Red}})_p \to G_{\text{Red}} \cdot p$. This defines a sheaf $j_*\mathscr{O}_{G/G_p}$ on $G \cdot p$ by the map $U \mapsto j_*\mathscr{O}_{G/G_p}(U) = \mathscr{O}_{G/G_p}(j^{-1}(U))$. Thus,

$$G \cdot p := (G_{\text{Red}} \cdot p, j_*\mathscr{O}_{G/G_p})$$

is a supermanifold. Recall that $\hat{p} = (\hat{\delta}_p, \hat{\delta}_p^*) : \mathscr{M} \to \mathscr{M}$ denotes the constant map at p. For each point $p \in M$, the *orbit map* of p is defined as a morphism of supermanifolds $\Psi_p = (\psi_p, \psi_p^*) : G \to M$ given by $\Psi \circ \langle \text{id}_G, \hat{p} \rangle$. This map satisfies $\Psi_p \circ R_g = \Psi_{g \cdot p}$ and $\psi(g) \circ \Psi_p = \Psi_p \circ L_g$ for every $g \in G$.

The action Ψ is deemed *transitive* if the underlying action of G_{Red} on M is transitive and the map $\mathfrak{g} \to T_pM$ defined as $X \mapsto X|_e \circ \Psi_p^* = \hat{X}|_p$ is surjective.

We say that a supermanifold is *G-homogeneous* if there exists a Lie supergroup G acting transitively on it. A Riemannian supermanifold $\mathscr{M}$ is *homogeneous* if its isometry group $I(M)$ acts transitively. If $\mathscr{M}$ is G-homogeneous, then the orbit map, $\Psi_p : G/G_p \to \mathscr{M}$ taking the coset gG_p to $\psi_p(g) \in G \cdot p$, is a diffeomorphism onto its image. Hence, a Riemannian homogeneous supermanifold $\mathscr{M}$ is diffeomorphic to a homogeneous superspace G/H, where $G \subset I(M)$ is closed and $H = G_p$ for some $p \in M$.

In the standard theory, a Riemannian metric on a G-homogeneous space is labelled G-invariant if each $g \in G$ acts by isometry. We extend this definition to graded Riemannian metrics on G-homogeneous supermanifolds: we say a graded Riemannian metric is *G-invariant* if each $g \in G_{\text{Red}}$ acts by isometry, and the image of $\mathfrak{g} \to \mathscr{T}_M(M)$ lies in the subalgebra of graded Killing fields.

The following result gives a characterisation of G-invariant graded Riemannian metrics, enabling us to operate at the Lie superalgebra level rather than the group level.

THEOREM 3.9. [Goe08, Theorem 3] *Let M be a G-homogeneous supermanifold, and fix $p \in M_{\text{Red}}$. There is a $1-1$ correspondence between G-invariant graded Riemannian metrics on $\mathscr{M}$ and Ad^{G_p}-invariant scalar superproducts on $\mathfrak{g}/\mathfrak{g}_\mathfrak{p} \cong T_pM$.*

We say that a homogeneous supermanifold G/H is *reductive* if the Lie superalgebra $\mathfrak{g}$ of G admits a decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$$

such that $\text{Ad}_h \mathfrak{m} \subset \mathfrak{m}$ for every $h \in H_{\text{Red}}$. In this case, we say that $\mathfrak{m}$ is Ad^H-invariant. If G/H is reductive, we see that $\mathfrak{m} \cong T_o(G/H)$.

PROPOSITION 3.10. *The isotropy representation of G/H is equivalent to the adjoint representation of H in $\mathfrak{m} \cong \mathfrak{g}/\mathfrak{g}_p$.*

PROOF. We need to show that $\text{Ad}^G(h)Y = \text{Ad}^{G/H}(h)Y$ for $h \in H$ and $Y \in \mathfrak{m}$. Indeed, for any $g \in (G_{\text{Red}})_p = H$ and any $X \in \mathfrak{g}$, we compute

$$(\text{Ad}^G(g)X)_e \circ \psi_p^* = X_e \circ L_g^* \circ R_{g^{-1}}^* \circ \psi_p^* = X_e \circ L_g^* \circ \psi_p^* = X_e \circ \psi_p^* \circ \psi(g)^* = \mathrm{d}\psi(g)(X_e \circ \psi_p^*).$$

The equivalence of representations at the level of Lie superalgebras can be seen via the relationship

$$[X,Y]_e \circ \psi_p^* = -\left[(Y_e \otimes I) \circ \psi^*, X_e \circ \psi_p^*\right].$$

□

CHAPTER 4

Ricci flow in the non-super setting

The Ricci flow is a geometric flow that describes the evolution of a Riemannian metric over time. We begin by discussing the motivation behind the Ricci flow, and exploring some classical results in the field. We then introduce the homogeneous Ricci flow, reviewing the main questions and recent progress in the field. At the end of the chapter, we narrow our focus to the super setting, explicitly computing the curvature and analysing the Ricci flow equation for the Lie supergroup $\mathsf{SL}(1|1)$.

4.1. An introduction to Ricci flow

Let (M, g_0) be a Riemannian manifold. The Ricci flow equation is the second-order weakly parabolic partial differential equation

$$\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Ric}(g(t)), \quad g(0) = g_0,$$

where $\{g(t)\}_{t\in[0,T)}$ is a one-parameter family of Riemannian metrics. It is not always possible to deform the metric $g(t)$ indefinitely. We say that the time T is a *singularity* in the Ricci flow if the flow cannot be smoothly extended past T. We will make this definition more precise shortly. For now, we are interested in techniques to overcome singularities.

Suppose that $g(t)$ is a Ricci flow for $t \in [0,T)$. We know that the Ricci tensor is scaling invariant; let us now see how the flow changes when scaling the metric. Define $\tilde{g}(x,t) := \lambda g\left(x, \frac{t}{\lambda}\right)$ for some $\lambda > 0$ and $t \in [0, \lambda T)$. Then,

$$\frac{\partial}{\partial t}\tilde{g}(x,t) = \frac{\partial}{\partial t} g\left(x, \frac{t}{\lambda}\right) = -2\,\mathrm{Ric}\, g\left(\frac{t}{\lambda}\right)(x) = -2\,\mathrm{Ric}\,\tilde{g}(t)(x).$$

The scaled metric $\tilde{g}$ is also a Ricci flow. This fact turns out to be quite useful in analysing singularities of the flow. For example, in the case the manifold shrinks to a point, as it does in Figure 1, one can rescale the metric to keep the volume constant over time. There are more technical ways to overcome singularities such as ‘surgery’, but this is beyond the scope of our project.

4.1.1. Variational formula.

Let h be a symmetric $(0,2)$-tensor. In this section we will compute the variation of a few quantities under some deformation of the metric tensor $\frac{\partial}{\partial t} g_{ij} = h_{ij}$. Upon setting $h = -2\,\mathrm{Ric}\, g$, we derive the Ricci flow equation. Consequently, the following results provide insights into the evolution of these quantities under the Ricci flow. In the following, we will use the fact that computations for tensors hold coordinate invariantly.

LEMMA 4.1. [Bes87, Theorem 1.174] *Let (M,g) be a Riemannian manifold and h a symmetric $(0,2)$-tensor. Then, the differentials at g, in the direction h, of various quantities are given by the following expressions:*

(i) The inverse of the metric g^{-1}

$$\frac{\partial}{\partial t} g^{ij} = -g^{ik} g^{jl} h_{kl};$$

(ii) the Levi-Civita connection

$$\frac{\partial}{\partial t}\Gamma^k_{ij} = \frac{1}{2}g^{kl}(\nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij});$$

(iii) the Riemann curvature tensor

$$\frac{\partial}{\partial t}R^l_{ijk} = \frac{1}{2}g^{lp}\left(\nabla_i\nabla_j h_{kp} + \nabla_i\nabla_k h_{jp} - \nabla_i\nabla_p h_{jk} - \nabla_j\nabla_i h_{kp} - \nabla_j\nabla_k h_{ip} + \nabla_j\nabla_p h_{ik}\right);$$

(iv) the Ricci tensor

$$\frac{\partial}{\partial t}R_{jk} = \frac{1}{2}g^{pq}\left(\nabla_q\nabla_j h_{kp} + \nabla_q\nabla_k h_{jp} - \nabla_q\nabla_p h_{jk} - \nabla_j\nabla_k h_{qp}\right);$$

(v) the scalar curvature

$$\frac{\partial}{\partial t}S = -\Delta \operatorname{tr}_g h + \nabla^p\nabla^q h_{pq} - \langle h, \mathrm{Ric}\rangle;$$

(vi) the volume element $\mathrm{d}\mu$

$$\frac{\partial}{\partial t}\mathrm{d}\mu = \frac{1}{2}\left(g^{ij}\frac{\partial}{\partial t}g_{ij}\right) = \frac{\operatorname{tr}_g h}{2}\mathrm{d}\mu.$$

PROOF. We only prove the first two as the general style of proof remains the same.

(i) By definition, $g^{ik}g_{kl} = \delta^i_l$. Considering the time derivative $\frac{\partial}{\partial t}g^{ik}g_{kl}$, we have

$$\left(\frac{\partial}{\partial t}g^{ik}\right)g_{kl} + g^{ik}\frac{\partial}{\partial t}g_{kl} = \left(\frac{\partial}{\partial t}g^{ik}\right)g_{kl} + g^{ik}h_{kl} = 0.$$

Solving for $\frac{\partial}{\partial t}g^{ij}$ we find

$$\frac{\partial}{\partial t}g^{ij} = \left(\frac{\partial}{\partial t}g^{ik}\right)g_{kl}g^{jl} = -g^{ik}g^{jl}h_{kl}.$$

(ii) Recall the formula for the Christoffel symbols in some local coordinate system $\{x_i\}$:

$$\Gamma^k_{ij} = \frac{1}{2}g^{kl}\left(\frac{\partial}{\partial x_i}g_{jl} + \frac{\partial}{\partial x_j}g_{il} - \frac{\partial}{\partial x_l}g_{ij}\right).$$

Hence,

$$\frac{\partial}{\partial t}\Gamma^k_{ij} = \frac{1}{2}\frac{\partial}{\partial t}g^k l\left(\frac{\partial}{\partial x_i}g_{jl} + \frac{\partial}{\partial x_j}g_{il} - \frac{\partial}{\partial x_l}g_{ij}\right) + \frac{1}{2}g^{kl}\left(\frac{\partial}{\partial x_i}\frac{\partial}{\partial t}g_j l + \frac{\partial}{\partial x_j}\frac{\partial}{\partial t}g_{il} - \frac{\partial}{\partial l}\frac{\partial}{\partial t}g_{ij}\right).$$

Choosing geodesic coordinates about a point $p \in M$, we have $\Gamma^k_{ij}(p) = 0$. Subsequently we find $\frac{\partial}{\partial x_i}A_{jk} = \nabla_i A_{jk}$ for any tensor A; in particular, $\frac{\partial}{\partial x_i}g_{jk} = 0$ for all i, j, k. Thus,

$$\frac{\partial}{\partial t}\Gamma^k_{ij}(p) = \frac{1}{2}g^{kl}\left(\nabla_i\frac{\partial}{\partial t}g_{jl} + \nabla_j\frac{\partial}{\partial t}g_{il} - \nabla_l\frac{\partial}{\partial t}g_{ij}\right)(p).$$

□

Importantly, the evolution of the scalar curvature has an invariant formula:

(4.1) $$\frac{\partial}{\partial t}S = -\Delta \operatorname{tr}_g h + \operatorname{div}(\operatorname{div} h) - \langle h, \mathrm{Ric}\rangle.$$

4.2. Homogeneous Ricci flow

In this section, we describe the current literature on the homogeneous Ricci flow. Let G be a compact Lie group and H a closed subgroup of G. Denote by $\mathfrak{h} \subset \mathfrak{g}$ the Lie algebras of G and H. Suppose that G acts almost effectively and consider the homogeneous space G/H.

4.2.1. The Setup. Fix a bi-invariant Riemannian metric on G. Let Q denote the induced Ad^G-invariant inner product on $\mathfrak{g}$. The Q-orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ is an Ad^H-invariant subspace, which we denote by $\mathfrak{m}$. Let o be the identity coset eH in G/H. We then identify $\mathfrak{m}$ with the tangent space $T_o(G/H)$, where H acts on $\mathfrak{m}$ via the adjoint representation Ad^H. Every G-invariant metric on G/H is determined by an Ad^H-invariant scalar product on $\mathfrak{g}/\mathfrak{h}$.

Consider a Q-orthogonal Ad^H-invariant decomposition

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \ldots \oplus \mathfrak{m}_s \tag{4.2}$$

In general, this decomposition is not unique unless the summands $\mathfrak{m}_i$ are irreducible and inequivalent.

Given a G-invariant Riemannian metric g on M, it is always possible to choose a decomposition of the form (4.2) such that g is diagonal with repsect to Q.

PROPOSITION 4.2. *The metric respects the splitting of $\mathfrak{m}$ into s irreducible, inequivalent summands; that is*

$$\langle \cdot, \cdot \rangle = x_1\, Q|_{\mathfrak{m}_1} \oplus \ldots \oplus x_s Q|_{\mathfrak{m}_s}, \tag{4.3}$$

where $x_i \in \mathbb{R} \setminus \{0\}$ for all $1 \leq i \leq s$.

REMARK 4.3. *Since we are dealing with Riemannian metrics, we can assume that $x_i > 0$ for all i.*

PROOF. We aim to apply Schur's lemma to the endomorphism $G : \mathfrak{m} \to \mathfrak{m}$ defined by

$$Q(G(X), Y) = g(X, Y).$$

Indeed, if we can show that G intertwines the isotropy representation $(\mathrm{d}L_h)_o$, where $h \in H$ and $o = eH$, then Schur's lemma implies that $G|_{\mathfrak{m}_i} = x_i \mathrm{Id}|_{\mathfrak{m}_i}$ as required. By the left-invariance of g, we find

$$Q(G(X), Y) = g(X, Y) = g((\mathrm{d}L_h)_o X, (\mathrm{d}L_h)_o Y) = Q(G((\mathrm{d}L_h)_o X), (\mathrm{d}L_h)_o Y)$$

for all X and Y. Hence,

$$Q(G(X), Y) = Q(G((\mathrm{d}L_h)_o X), (\mathrm{d}L_h)_o Y) = Q((\mathrm{d}L_h)_o^T G((\mathrm{d}L_h)_o X), Y).$$

By orthogonality, $(\mathrm{d}L_h)_o^T = (\mathrm{d}L_h)_o^{-1}$ and we can conclude that

$$(\mathrm{d}L_h)_o G(X) = G((\mathrm{d}L_h)_o X).$$

Hence, G is an intertwiner. □

PROPOSITION 4.4. *The Ricci tensor respects the splitting of $\mathfrak{m}$ into s irreducible, inequivalent summands:*

$$\mathrm{Ric}(g) = r_1 x_1\, Q|_{\mathfrak{m}_1} \oplus \ldots \oplus r_s x_s\, Q|_{\mathfrak{m}_s}.$$

PROOF. We again aim to apply Schur's lemma, now to the Ricci endomorphism:

$$g(\mathrm{ric}(\cdot), \cdot) = \mathrm{Ric}(g)(\cdot, \cdot).$$

Since left-translation L_h is a diffeomorphism and the Ricci tensor is isometry invariant,

$$g(\mathrm{ric}(\mathrm{d}(L_h)_o X), \mathrm{d}(L_h)_o Y) = \mathrm{Ric}(g)(\mathrm{d}(L_h)_o X, \mathrm{d}(L_h)_o Y) = (L_h)^* \mathrm{Ric}(g)(X, Y)$$

$$= (L_h)^* g(\mathrm{ric}(X),Y) = g(\mathrm{d}(L_h)_o\,\mathrm{ric}(X), \mathrm{d}(L_h)_o Y),$$

for all X,Y. That is, $\mathrm{ric}(\cdot)$ intertwines the isotropy representation. Schur's lemma then implies $\mathrm{Ric}(g) = \sum_{i=1}^s r_i\, g|_{m_i}$. Combining this with the previous proposition gives the required expression. □

PROPOSITION 4.5. *The square norm of the Riemann curvature tensor respects the splitting of the metric:*

$$|\mathrm{Rm}(g(t))|^2_{g(t)} = \sum_{i=1}^s |\mathrm{Rm}(g(t)|_{\mathfrak{m}_i})|^2_{g(t)|_{\mathfrak{m}_i}} = \sum_{i=1}^s \frac{1}{x_i(t)^2} |\mathrm{Rm}(Q|_{\mathfrak{m}_i})|^2_{Q|_{\mathfrak{m}_i}}.$$

PROOF. The proof is similar to that of Propositions 4.2 and 4.4. □

Let B denote the Killing form on the Lie algebra $\mathfrak{g}$. As each $\mathfrak{m}_i$ is irreducible, there exists $b_i \geq 0$ such that

$$B|_{\mathfrak{m}_i} = -b_i Q|_{\mathfrak{m}_i}.$$

We define the structure constants

$$[ijk] = \sum_{\alpha,\beta,\gamma} Q([e_\alpha, e_\beta], e_\gamma)^2,$$

where $\{e_\alpha\}, \{e_\beta\}$, and $\{e_\gamma\}$ are Q-orthonormal bases of $\mathfrak{m}_i, \mathfrak{m}_j$, and $\mathfrak{m}_k$. It is clear that $[ijk]$ is symmetric in all three indices since Q is $\mathrm{ad}_{\mathfrak{g}}$-invariant.

THEOREM 4.6. *Let G/H be a compact homogeneous space with decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}_1 \oplus \ldots \oplus \mathfrak{m}_s$. Define a G-invariant metric g on G/H as in (4.3). The Ricci curvature tensor of g on each $\mathfrak{m}_i$ is given by*

$$r_i x_i = \frac{b_i}{2} - \frac{1}{2d_i} \sum_{j,k=1}^s \frac{x_k}{x_j}[ijk] + \frac{1}{4d_i} \sum_{j,k=1}^s \frac{x_i^2}{x_j x_k}[ijk],$$

where $d_i = \dim \mathfrak{m}_i$.

PROOF. This is a lengthy computation and can be found in [PR19, Lemma 3.3]. □

The scalar curvature is given by the trace of the Ricci tensor: $S(g) = \sum_{i=1}^s r_i d_i$. In light of the above theorem, a solution $g(t)$ to the Ricci flow equation on G/H must satisfy the system of ODEs

$$\frac{\partial}{\partial t} x_i(t) = -2 r_i x_i(t),$$

where $x_i(t) > 0$ for $1 \leq i \leq s$.

4.2.2. Singularities – Type I and Type II. Let (M,g) be a closed Riemannian manifold. A solution $g(t)$ to the Ricci flow on $M \times [0,\infty)$ is a *maximal solution* if $|\mathrm{Rm}(g(t))(x,t)|$ is unbounded as $t \to T$. We call the maximal solution *singular* if, in addition, $T < \infty$. Singular solutions to the Ricci flow can be classified into two types. A solution develops a *type I singularity* at $t = T$ if

(i) $T < \infty$,

(ii) $\sup_{t\in[0,T)} \left(\sup_{p\in M} |\mathrm{Rm}(g(t))|_{g(t)}(p,t) \right) = \infty$, and

(iii) $\sup_{t\in[0,T)} \left((T-t) \sup_{p\in M} |\mathrm{Rm}(g(t))|_{g(t)}(p,t) \right) < \infty$.

A solution develops a *type II singularity* at $t = T$ if, in addition to (i) and (ii),

$$\sup_{t\in[0,T)} \left((T-t) \sup_{p\in M} |\mathrm{Rm}(g(t))|_{g(t)}(p,t) \right) = \infty.$$

REMARK 4.7. *As we mainly deal with homogeneous spaces, we can ignore the supremum over M in the above definitions.*

4.2.3. A Different Type of Convergence. The goal of this section is to understand what it means for a sequence of Riemannian manifolds to converge. We focus on the weakest type of convergence available: Gromov-Hausdorff convergence. The Gromov-Hausdorff distance is a way of measuring distances between metric spaces; in essence, it measures how far two compact metric spaces are from being isometric. Let (X, d_X) and (Y, d_Y) be two metric spaces. We say that there exists an ε-approximation between X and Y if, for $S \subset X \times Y$, two conditions hold:

(i) the projections $\mathrm{pr}_1 : S \to X$ and $\mathrm{pr}_2 : S \to Y$ are onto;
(ii) for all $(x_1, y_1), (x_2, y_2) \in S$, $|d_X(x_1, x_2) - d_Y(y_1, y_2)| < \varepsilon$.

If there exists an ε-approximation between X and Y, we write $X \sim_\varepsilon Y$. We then define the *Gromov-Hausdorff distance* between X and Y to be

$$d_{G-H}(X,Y) = \inf\{\varepsilon : X \sim_\varepsilon Y\}.$$

If no such ε exists then $d_{G-H}(X,Y) = \infty$. Given a sequence of metric spaces $\{(X_n, d_{X_n})\}_n$, we say that $(X_n, d_{X_n}) \to (X, d)$ in the Gromov-Hausdorff topology if $d_{G-H}(X_n, X) \to 0$ as $n \to \infty$. Buzano proved the following useful result.

THEOREM 4.8. [Buz14, Proposition 2.6] *Let G/H be a compact and connected homogeneous space. Suppose there exists an intermediate Lie group K, with $H < K < G$. Let $\mathfrak{g}, \mathfrak{h}, \mathfrak{k}$ be the Lie algebras of G, H and K respectively. Suppose $\mathfrak{k}$ is Ad^H-invariant and that every G-invariant Riemannian metric on G/H is a submersion metric*

$$K/H \to G/H \to G/K.$$

Then, if the fibre K/H shrinks to a point, G/H converges in the Gromov-Hausdorff sense to G/K.

4.2.4. The story so far. Einstein metrics are of great interest; they are the metrics of constant Ricci curvature. A natural question is whether a manifold admits an Einstein metric, and if so, is it unique. In the homogeneous setting, this problem has been studied in depth.

Wang and Ziller [WZ86] study the existence and non-existence of homogeneous Einstein metrics by considering a variational interpretation. More precisely, let $\mathscr{M}$ denote the set of Riemannian metrics on a compact manifold M. The *total scalar curvature functional* $\mathscr{E} : \mathscr{M} \to \mathbb{R}$ is given by

$$\mathscr{E}(g) = \int_M S(g) \mathrm{dvol}_g,$$

where $\mathrm{vol}_g = \sqrt{|\det(g_{ij})|}$. This is homogeneous of degree $\frac{n}{2}$, and so we often restrict to the space of volume one metrics $\mathscr{M}_1$.

PROPOSITION 4.9. [Bes87, Proposition 4.17] *Given a compact Riemannian manifold (M, g), the functional $\mathscr{E}$ has first variation given by*

$$\mathrm{d}\mathscr{E}_g(h) = \int_M \left\langle \frac{S}{2} g - \mathrm{Ric}\, g, h \right\rangle_g \mathrm{dvol}_g,$$

for some $(0,2)$-tensor h.

PROOF. Using 4.1,

$$\mathrm{d}\mathscr{E}_g(h) = \int_M \Delta_g \operatorname{tr}_g h + \operatorname{div}(\operatorname{div} h) - \langle \operatorname{Ric} g, h\rangle + \frac{S}{2}\operatorname{tr}_g h \mathrm{dvol}_g.$$

Stokes' theorem then implies

$$\mathrm{d}\mathscr{E}_g(h) = \int_M \frac{S}{2}\langle g, h\rangle - \langle \operatorname{Ric} g, h\rangle \,\mathrm{dvol}_g = \int_M \left\langle \frac{S}{2} g - \operatorname{Ric} g, h \right\rangle \mathrm{dvol}_g.$$

□

As a consequence of the above computation, we see that critical points of $\mathscr{E}$ are Einstein metrics of volume 1 with constant $\frac{S}{2}$. Wang and Ziller use this characterisation to prove the following result.

THEOREM 4.10. [WZ86, Theorem 1] *Let G be a connected compact Lie group and H a connected closed subgroup such that G/H is effective. Then, the scalar curvature functional $\mathscr{E}$ on the set of G-invariant metrics with volume* 1 *is bounded from above and proper if and only if H is a maximal connected subgroup of G. For such a G/H, $\mathscr{E}$ assumes its global maximum at a G-invariant Einstein metric.*

Further existence theorems for homogeneous Einstein metrics will involve searching for saddle points of $\mathscr{E}$ when it is unbounded from above and below. Böhm, Wang and Ziller [BWZ04] consider this.

The study of Einstein metrics is intimately linked to the Ricci flow. The long time behaviour of homogeneous Ricci flows has been studied by numerous authors. Buzano [Buz14] completely classifies the behaviour for compact homogeneous spaces G/H where the isotropy representation decomposes into two inequivalent, irreducible summands: $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$. It was shown that a Type I singularity is always reached in finite time. Depending on the subgroup H, Buzano found different behaviour as the singular time was approached. In the case where there exists an intermediate subgroup $H < K < G$ such that its Lie algebra is Ad^H-invariant, Buzano showed that either G/H would shrink to a point or converge in the Gromov-Hausdorff topology to G/K.

Extending these results, Böhm proved the following.

THEOREM 4.11. [Böh15, Theorem 1] *A homogeneous Ricci flow with finite extinction time develops a Type I singularity.*

THEOREM 4.12. [Böh15, Theorem 2] *Let M^n be a compact homogeneous space not diffeomorphic to the torus T^n. Then any homogeneous Ricci flow solution has finite extinction time.*

THEOREM 4.13. [Böh15, Theorem 3] *Let $M^n = G/H$ be a compact homogeneous space not diffeomorphic to the torus T^n. Suppose that the isotropy representation decomposes into pairwise inequivalent summands. Then for any homogeneous Ricci flow on G/H there exists a compact intermediate subgroup K, such that $E_\infty = K/H$.*

In the above, E_∞ is the compact homogeneous space appearing in the limit.

4.3. A motivating example in the super setting

In this section, we study the Ricci flow of left-invariant metrics on the Lie supergroup $\mathsf{SL}(1|1)$. The Lie supergroup $\mathsf{SL}(1|1)$ has the associated sHCp $(\mathsf{SL}(1) \times \mathsf{SL}(1) \times \mathbb{R}, \mathfrak{sl}(1|1))$. Let

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ Z = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

denote the standard basis of $\mathfrak{sl}(1|1)$. These basis elements satisfy the following commutation relations:

$$[X,Y] = [X,Z] = 0, \quad [Y,Z] = X.$$

Equip $\mathsf{SL}(1|1)$ with a left-invariant, graded Riemannian metric g. This induces a scalar superproduct $\langle \cdot, \cdot \rangle$ on $\mathfrak{sl}(1|1)$. Since g is symmetric on $\mathsf{SL}(1|1)_0$ and symplectic on $\mathsf{SL}(1|1)_1$, we may assume g takes the form

$$g = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & 0 & x_{23} \\ x_{31} & -x_{23} & 0 \end{pmatrix},$$

where $x_{ij} \in \mathbb{R} \setminus \{0\}$ for all i, j.

LEMMA 4.14. *There exists a linear transformation $T : \mathfrak{sl}(1|1) \to \mathfrak{sl}(1|1)$ that preserves the bracket relations and is such that g takes the form*

$$\begin{pmatrix} x_1 & 0 & 0 \\ 0 & 0 & x_2 \\ 0 & -x_2 & 0 \end{pmatrix}, \tag{4.4}$$

where $x_1, x_2 \in \mathbb{R} \setminus \{0\}$.

PROOF. Define S by the change of basis that maps $\{X,Y,Z\}$ to $\left\{X, Y - \frac{x_{21}}{x_{11}}X, Z - \frac{x_{31}}{x_{11}}X\right\} =: \{X_1, X_2, X_3\}$. Under this change of basis, g becomes

$$\tilde{g} := P^T g P = \begin{pmatrix} x_{11} & x_{12} - x_{21} & x_{13} - x_{31} \\ 0 & -\frac{x_{21}x_{12}}{x_{11}} & \frac{x_{11}x_{23} - x_{21}x_{13}}{x_{11}} \\ 0 & \frac{-x_{11}x_{23} - x_{31}x_{12}}{x_{11}} & -\frac{x_{31}x_{13}}{x_{11}} \end{pmatrix},$$

where P is the matrix representation of S. We know that $\tilde{g}$ must remain symplectic on $\mathsf{SL}(1|1)_1$, so $x_{31}x_{13} = x_{21}x_{12} = 0$ and $\frac{x_{11}x_{23} - x_{21}x_{13}}{x_{11}} = \frac{x_{11}x_{13} + x_{31}x_{12}}{x_{11}}$. Some messy algebraic manipulation gives (4.4). It stands to verify that the bracket relations are preserved under S. Indeed,

$$\begin{aligned}
&[X_i, X_i] = 0, \quad i = 1,2,3 \\
&[X_1, X_2] = \left[X, Y - \frac{x_{21}}{x_{11}}X\right] = 0 = [X_2, X_1], \\
&[X_1, X_3] = \left[X, Z - \frac{x_{31}}{x_{11}}\right] = 0 = [X_3, X_1], \\
&[X_2, X_3] = \left[Y - \frac{x_{21}}{x_{11}}X, Z - \frac{x_{31}}{x_{11}}X\right] = [Y,Z] - \frac{x_{31}}{x_{11}}[Y,X] - \frac{x_{21}}{x_{11}}[X,Z] + \frac{x_{21}x_{31}}{x_{11}^2}[X,X] = X = [X_3, X_2].
\end{aligned}$$

□

Fix a basis $\{X_1, X_2, X_3\}$ of $\mathfrak{sl}(1|1)$ such that g resembles (4.4). Since $\mathfrak{sl}(1|1)$ is finite-dimensional, we may identify the dual space $\mathfrak{sl}(1|1)^*$ with $\mathfrak{sl}(1|1)$. The dual basis

$$\{X^1, X^2, X^3\} = \left\{\frac{1}{x_1}X_1, \frac{1}{x_2}X_3, -\frac{1}{x_2}X_2\right\}$$

is such that $\langle X_i, X^j \rangle = \delta_{ij}$. Fix ∇ to be the Levi-Civita connection. We now compute the Ricci curvature of g. For this, we will compute the covariant derivatives $\nabla_{X_i}X_j$, where $i, j = 1,2,3$.

We start by computing $\nabla_{X_i} X_j$ for all $i, j = 1,2,3$. By the left-invariance of X_1, X_2 and X_3, the first three terms in the right hand side of (3.2) vanish. Since ∇ is torsion free, we find

$$\nabla_{X_1} X_2 = \nabla_{X_2} X_1, \quad \nabla_{X_1} X_3 = \nabla_{X_3} X_1, \text{ and } \nabla_{X_2} X_3 = X_1 - \nabla_{X_3} X_2.$$

It is easy to see $\nabla_{X_i} X_i = 0$ for $i = 1,2,3$ and so we need only compute $\nabla_{X_1} X_2, \nabla_{X_1} X_3$ and $\nabla_{X_2} X_3$. Firstly,

$$\begin{aligned}
\nabla_{X_1} X_2 &= \langle \nabla_{X_1} X_2, X^1 \rangle X_1 + \left\langle \nabla_{X_1} X_2, X^2 \right\rangle X_2 + \left\langle \nabla_{X_1} X_2, X^3 \right\rangle X_3, \\
\left\langle \nabla_{X_1} X_2, X^1 \right\rangle &= \frac{1}{2x_1} \left(\langle [X_1, X_2], X_1 \rangle - \langle [X_2, X_1], X_1 \rangle + \langle [X_1, X_1], X_2 \rangle \right) = 0, \\
\left\langle \nabla_{X_1} X_2, X^2 \right\rangle &= \frac{1}{2x_2} \left(\langle [X_1, X_2] X_3 \rangle - \langle [X_2, X_3], X_1 \rangle - \langle [X_3, X_1], X_2 \rangle \right) = -\frac{x_1}{2x_2}, \\
\left\langle \nabla_{X_1} X_2, X^3 \right\rangle &= -\frac{1}{2x_2} \left(\langle [X_1, X_2], X_2 \rangle - \langle [X_2, X_2], X_1 \rangle - \langle [X_2, X_1], X_1 \rangle \right) = 0, \\
\implies \nabla_{X_1} X_2 &= \nabla_{X_2} X_1 = -\frac{x_1}{2x_2} X_2.
\end{aligned}$$

Now for $\nabla_{X_1} X_3 = \nabla_{X_3} X_1$:

$$\begin{aligned}
\nabla_{X_1} X_3 &= \left\langle \nabla_{X_1} X_3, X^1 \right\rangle + \left\langle \nabla_{X_1} X_3, X^2 \right\rangle + \left\langle \nabla_{X_1} X_3, X^3 \right\rangle, \\
\left\langle \nabla_{X_1} X_3, X^1 \right\rangle &= \frac{1}{2x_1} \left(\langle [X_1, X_3], X_1 \rangle - \langle [X_3, X_1], X_1 \rangle + \langle [X_1, X_1], X_3 \rangle \right) = 0, \\
\left\langle \nabla_{X_1} X_3, X^2 \right\rangle &= \frac{1}{2x_2} \left(\langle [X_1, X_3], X_3 \rangle - \langle [X_3, X_3], X_1 \rangle - \langle [X_3, X_1], X_3 \rangle \right) = 0, \\
\left\langle \nabla_{X_1} X_3, X^3 \right\rangle &= -\frac{1}{2x_2} \left(\langle [X_1, X_3], X_2 \rangle - \langle [X_3, X_2], X_1 \rangle - \langle [X_2, X_3], X_3 \rangle \right) = \frac{x_1}{2x_2}, \\
\implies \nabla_{X_1} X_3 &= \nabla_{X_3} X_1 = \frac{x_1}{2x_2} X_3.
\end{aligned}$$

Finally, for $\nabla_{X_2} X_3 = X_1 - \nabla_{X_3} X_2$:

$$\begin{aligned}
\nabla_{X_2} X_3 &= \left\langle \nabla_{X_2} X_3, X^1 \right\rangle + \left\langle \nabla_{X_2} X_3, X^2 \right\rangle + \left\langle \nabla_{X_2} X_3, X^3 \right\rangle, \\
\left\langle \nabla_{X_2} X_3, X^1 \right\rangle &= \frac{1}{2x_1} \left(\langle [X_2, X_3], X_1 \rangle + \langle [X_3, X_1], X_2 \rangle + \langle [X_1, X_2], X_3 \rangle \right) = \frac{1}{2}, \\
\left\langle \nabla_{X_2} X_3, X^2 \right\rangle &= \frac{1}{2x_2} \left(\langle [X_2, X_3], X_3 \rangle - \langle [X_3, X_3], X_2 \rangle + \langle [X_3, X_2], X_3 \rangle \right) = 0, \\
\left\langle \nabla_{X_2} X_3, X^3 \right\rangle &= -\frac{1}{2x_2} \left(\langle [X_2, X_3], X_2 \rangle - \langle [X_3, X_2], X_2 \rangle + \langle [X_2, X_2], X_3 \rangle \right) = 0, \\
\implies \nabla_{X_2} X_3 &= \nabla_{X_3} X_2 = \frac{1}{2} X_1.
\end{aligned}$$

In our basis $\{X_1, X_2, X_3\}$, (3.4) amounts to

$$\operatorname{Ric}(X_i, X_j) = \sum_{k=1}^{3} (-1)^{|k|(|j|+1)} \left\langle R(X_i, X_k) X_j, X^k \right\rangle,$$

where $R(X_i, X_k) X_j := [\nabla_{X_i}, \nabla_{X_k}] X_j - \nabla_{[X_i, X_k]} X_j$. One can easily see that the only non-zero components of the Ricci tensor are $\operatorname{Ric}(X_1, X_1), \operatorname{Ric}(X_2, X_3)$ and $\operatorname{Ric}(X_3, X_2)$. By the symmetry of the Ricci tensor, we need only compute $\operatorname{Ric}(X_1, X_1)$ and $\operatorname{Ric}(X_2, X_3)$.

We first compute $\mathrm{Ric}(X_1,X_1)$:

$$\begin{aligned}
\mathrm{Ric}(X_1,X_1) &= \left\langle R(X_1,X_1)X_1,X^1\right\rangle - \left\langle R(X_1,X_2)X_1,X^2\right\rangle - \left\langle R(X_1,X_3)X_1,X^3\right\rangle \\
&= \frac{1}{x_1}\langle[\nabla_{X_1},\nabla_{X_1}]X_1,X_1\rangle - \frac{1}{x_2}\langle\nabla_{X_1}\nabla_{X_2}X_1,X_3\rangle + \frac{1}{x_2}\langle\nabla_{X_1}\nabla_{X_3}X_1,X_2\rangle \\
&= -\frac{1}{x_2}\left\langle -\frac{x_1}{2x_2}\nabla_{X_1}X_2,X_3\right\rangle + \frac{1}{x_2}\left\langle \frac{x_1}{2x_2}\nabla_{X_1}X_3,X_2\right\rangle \\
&= \frac{x_1}{2x_2^2}\left\langle -\frac{x_1}{2x_2}X_2,X_3\right\rangle + \frac{x_1}{2x_2^2}\left\langle \frac{x_1}{2x_2}X_3,X_2\right\rangle \\
&= -\frac{x_1^2}{4x_2^3}\langle X_2,X_3\rangle + \frac{x_1^2}{4x_2^3}\langle X_3,X_2\rangle = -\frac{x_1^2}{4x_2^2} - \frac{x_1^2}{4x_2^2} = -\frac{x_1^2}{2x_2^2}.
\end{aligned}$$

For $\mathrm{Ric}(X_2,X_3) = -\mathrm{Ric}(X_3,X_2)$, we find

$$\begin{aligned}
\mathrm{Ric}(X_2,X_3) &= \left\langle R(X_2,X_1)X_3,X^1\right\rangle + \left\langle R(X_2,X_2)X_3,X^2\right\rangle + \left\langle R(X_2,X_3)X_3,X^3\right\rangle \\
&= \frac{1}{x_1}\langle\nabla_{X_2}\nabla_{X_1}X_3 - \nabla_{X_1}\nabla_{X_2}X_3,X_1\rangle + \frac{1}{x_2}\langle[\nabla_{X_2},\nabla_{X_2}]X_3,X_3\rangle \\
&\quad - \frac{1}{x_2}\langle\nabla_{X_3}\nabla_{X_2}X_3 - \nabla_{X_1}X_3,X_2\rangle \\
&= \frac{1}{x_1}\left\langle \frac{x_1}{2x_2}\nabla_{X_2}X_3 - \frac{1}{2}\nabla_{X_1}X_1,X_1\right\rangle + \frac{1}{x_2}\langle 2\nabla_{X_2}\nabla_{X_2}X_3,X_3\rangle \\
&\quad - \frac{1}{x_2}\left\langle \frac{1}{2}\nabla_{X_3}X_1 - \frac{x_1}{2x_2}X_3,X_2\right\rangle \\
&= \frac{1}{4x_2^2}\langle X_1,X_1\rangle + \frac{1}{x_2}\langle\nabla_{X_2}X_1,X_3\rangle - \frac{1}{x_2}\left\langle \frac{x_1}{4x_2}X_3 - \frac{x_1}{2x_2}X_3,X_2\right\rangle \\
&= \frac{x_1}{4x_2} - \frac{x_1}{2x_2^2}\langle X_2,X_3\rangle + \frac{x_1}{4x_2^2}\langle X_3,X_2\rangle = -\frac{x_1}{2x_2}.
\end{aligned}$$

4.3.1. The Ricci flow equation. We now analyse the Ricci Flow equations for $\mathsf{SL}(1|1)$. It suffices to study the system of ODEs:

$$\begin{aligned}
\frac{\partial}{\partial t}x_1(t) &= \frac{x_1(t)^2}{x_2(t)^2}, \\
\frac{\partial}{\partial t}x_2(t) &= \frac{x_1(t)}{x_2(t)},
\end{aligned} \tag{4.5}$$

with initial data $x_1(0),x_2(0) \in \mathbb{R}\setminus\{0\}$. We claim that $\Psi(t,x_1,x_2) := \frac{x_1(t)}{x_2(t)}$ is a first integral of (4.5). Indeed, if $x_1(t)$ and $x_2(t)$ solve (4.5) then

$$\frac{\partial}{\partial t}\Psi(t,x_1,x_2) = \frac{x_2(t)\frac{\partial}{\partial t}x_1(t) - x_1(t)\frac{\partial}{\partial t}x_2(t)}{x_2(t)^2} = \frac{x_2(t)\left(\frac{x_1(t)^2}{x_2(t)^2}\right) - x_1(t)\left(\frac{x_1(t)}{x_2(t)}\right)}{x_2(t)^2} = 0.$$

Hence, solutions of (4.5) have the form $x_1(t) = \lambda x_2(t)$, for $\lambda = \frac{x_1(0)}{x_2(0)}$. That is,

$$x_1(t) = \lambda^2 t + x_1(0), \quad x_2(t) = \lambda t + x_2(0).$$

We must consider four cases:

(i) $x_1(0) > 0$ and $x_2(0) > 0$,
(ii) $x_1(0) > 0$ and $x_2(0) < 0$,
(iii) $x_1(0) < 0$ and $x_2(0) > 0$, and
(iv) $x_1(0) < 0$ and $x_2(0) < 0$.

It is easy to see that when $x_1(0) > 0$, renormalising by $\frac{1}{t}$ gives $x_1(t) \to \lambda^2$ and $x_2(t) \to \lambda$, as $t \to \infty$. This leaves cases (iii) and (iv).

Case (iii)*:* Since $\lambda > 0$, $\frac{\partial}{\partial t}x_1(t) = \lambda^2 > 0$ and $\frac{\partial}{\partial t}x_2(t) = \lambda > 0$. Thus, there exists $T := -\frac{x_2(0)^2}{x_1(0)} < \infty$ such that $x_1(T) = x_2(T) = 0$.

Case (iv)*:* Similarly, $\frac{\partial}{\partial t}x_1(t) = \lambda^2 > 0$ and $\frac{\partial}{\partial t}x_2(t) = \lambda < 0$ and so $T := -\frac{x_2(0)^2}{x_1(0)} < \infty$ has $x_1(T) = x_2(T) = 0$.

In these last two cases, we must stop the flow as the metric becomes degenerate. It is interesting to note that renormalising here does not improve the situation. The above discussion leads to the following result.

THEOREM 4.15. *Let g_0 be a left-invariant metric defined by* (4.4) *on* $\mathsf{SL}(1|1)$. *There exists a unique solution to the Ricci flow equation, and we observe one of two possible behaviours:*

(i) given $x_1(0) > 0$, the Ricci flow converges to an Einstein metric when renormalised by $\frac{1}{t}$;
(ii) given $x_1(0) < 0$, the Ricci flow has finite time extinction and the manifold shrinks to a point.

CHAPTER 5

The Ricci flow of homogeneous supermanifolds

Let $G = (G_{\mathrm{Red}}, \mathfrak{g})$ be a connected Lie supergroup and $H = (H_{\mathrm{Red}}, \mathfrak{h})$ be a closed connected Lie sub-supergroup. Assume that $\mathfrak{g}$ is a basic classical Lie superalgebra [Kac77], i.e., $\mathfrak{g}_0$ is reductive and $\mathfrak{g}$ admits a non-degenerate even supersymmetric bilinear form Q, which is $\mathrm{ad}_{\mathfrak{g}}$-invariant. Since G is connected, Q is Ad^G-invariant.

Recall that G-invariant metrics on the homogeneous superspace G/H are in one-to-one correspondence with Ad^H-invariant scalar superproducts on $T_o(G/H) \cong \mathfrak{g}/\mathfrak{h}$, where $o = H$ is the identity coset in G/H. In this chapter, we study the super homogeneous Ricci flow of G-invariant metrics on compact homogeneous superspaces G/H (recall that G/H is compact when the Lie algebra of G is a compact real form). We focus on the cases where the isotropy representation decomposes into one or two inequivalent irreducible summands. In section 5.6, we adapt techniques from Buzano [Buz14, Theorem 3.4] and [Buz12, Chapter 3], who extensively examines this problem in the non-super context.

5.1. Preliminaries

The setup is similar to the non-super setting. For completeness, we clarify the details here. Fix a Ad^H-invariant Q orthogonal complement $\mathfrak{m}$ of $\mathfrak{h}$ such that

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}. \tag{5.1}$$

Let g be a G-invariant metric on G/H (recall that this requires the odd dimension of $\mathfrak{m}$ to be even). The metric g affords us a Q-orthogonal Ad^H-invariant decomposition

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \ldots \oplus \mathfrak{m}_s \tag{5.2}$$

such that each $\mathfrak{m}_i$ is irreducible. In general, this decomposition is not unique. We assume that $\mathfrak{m}_i \not\cong \mathfrak{m}_j$ as $\mathrm{ad}_{\mathfrak{h}}$-representations whenever $i \neq j$. In this case, the summands are determined uniquely up to order.

As H is connected and g is G-invariant, there is an associated $\mathrm{ad}_{\mathfrak{h}}$-invariant even non-degenerate supersymmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{m}$. Schur's lemma implies that there exist $x_i \in \mathbb{R} \setminus \{0\}, 1 \leq i \leq s$ such that

$$\langle \cdot, \cdot \rangle = x_1 Q|_{\mathfrak{m}_1} \oplus \cdots \oplus x_s Q|_{\mathfrak{m}_s}, \tag{5.3}$$

where $x_i \in \mathbb{R} \setminus \{0\}$ for $1 \leq i \leq s$. Let $\mathrm{ric}(\cdot) : \mathfrak{m} \to \mathfrak{m}$ denote the Ricci endomorphism, defined by

$$\mathrm{Ric}(g)(\cdot, \cdot) := g(\mathrm{ric}(\cdot), \cdot).$$

With the same proof as Propositions 4.4 and 4.5, the Ricci curvature and square norm of the Riemann curvature respect the splitting of $\mathfrak{m}$ into irreducible summands:

$$\mathrm{Ric}(g) = r_1 x_1 Q|_{\mathfrak{m}_1} \oplus \ldots \oplus r_s x_s Q|_{\mathfrak{m}_s},$$

$$|\mathrm{Rm}(g(t))|^2_{g(t)} = \sum_{i=1}^{s} |\mathrm{Rm}(g(t)|_{\mathfrak{m}_i})|^2_{g(t)|_{\mathfrak{m}_i}} = \sum_{i=1}^{s} \frac{1}{x_i(t)^2} |\mathrm{Rm}(Q|_{\mathfrak{m}_i})|^2_{Q|_{\mathfrak{m}_i}},$$

where $\mathrm{ric}(\cdot) = r_i \mathrm{id}|_{\mathfrak{m}_i}(\cdot)$.

For each $\mathfrak{m}_i$, fix a Q-normalised basis $\{e^i_\alpha : 1 \le \alpha \le \dim \mathfrak{m}_i\}$ and let $\{\varepsilon^i_\alpha : 1 \le \alpha \le \dim \mathfrak{m}_i\}$ be a right dual basis such that $Q\left(e^i_\alpha, \varepsilon^i_{\alpha'}\right) = \delta_{\alpha\alpha'}$ for $1 \le \alpha, \alpha' \le \dim \mathfrak{m}_i$. Denote by I^i the index set of the basis for $\mathfrak{m}_i$. For every triple $1 \le i, j, k \le s$, define the structure constants of G/H by

$$[ijk] = \sum_{\alpha \in I^i, \beta \in I^j, \gamma \in I^k} -Q\left(e^k_\gamma, \left[e^i_\alpha, e^j_\beta\right]\right) Q\left(\left[\varepsilon^j_\beta, \varepsilon^i_\alpha\right], \varepsilon^k_\gamma\right), \tag{5.4}$$

The Ad^G-invariance of Q implies that $[ijk]$ is symmetric in all three indices.

REMARK 5.1. *In the classical setting, the structure constants are non-negative because of the squared term appearing in the sum. In this case, the structure constants can take any value in $\mathbb{R}$. The negative sign appears so that our expression for the Ricci curvature tensor agrees with the non-super case.*

Let $B(\cdot,\cdot)$ denote the Killing form of the Lie superalgebra $\mathfrak{g}$. Since $\mathfrak{m}_i$ is irreducible for each $1 \le i \le s$, Schur's lemma implies that there exist $b_i \in \mathbb{R}$ such that

$$B|_{\mathfrak{m}_i} = -b_i Q|_{\mathfrak{m}_i}.$$

PROPOSITION 5.2. [GPRZ23, Proposition 4.15] *With the above notation and assumptions, the Ricci tensor on each summand $\mathfrak{m}_i$ with $d_i := \mathrm{sdim}(\mathfrak{m}_i) \neq 0$ is given by*

$$r_i x_i = \frac{b_i}{2} - \frac{1}{2d_i} \sum_{j,k=1}^{s} \frac{x_k}{x_j}[ijk] + \frac{1}{4d_i} \sum_{j,k=1}^{s} \frac{x_i^2}{x_k x_j}[ijk]. \tag{5.5}$$

PROPOSITION 5.3. [GPRZ23, Proposition 4.16] *With the above notation and assumptions, the scalar curvature of g is given by*

$$S(g) = \sum_{i=1}^{s} r_i d_i = \frac{1}{2} \sum_{i=1}^{s} \frac{b_i d_i}{x_i} - \frac{1}{4} \sum_{i,j,k=1}^{s} \frac{x_k}{x_i x_j}[ijk]. \tag{5.6}$$

The following result follows from the above discussion.

THEOREM 5.4. *Let G/H be a Riemannian homogeneous supermanifold. A one-parameter family of metrics $g(t)$ defined by (5.3) is a solution to the homogeneous super Ricci flow if and only if $x_1(t), \cdots, x_s(t)$ satisfy the system*

$$\frac{\partial}{\partial t} x_i(t) = -b_i + \frac{1}{d_i} \sum_{j,k=1}^{s} \frac{x_k(t)}{x_j(t)}[ijk] - \frac{1}{2d_i} \sum_{j,k=1}^{s} \frac{x_i(t)^2}{x_k(t) x_j(t)}[ijk].$$

REMARK 5.5. *When there is no ambiguity, we sometimes omit writing x_i explicitly as a function of t.*

5.2. A variational approach

Hamilton [Ham82] showed that for a compact three-dimensional Riemannian manifold (M, g) with $\mathrm{Ric}(g) > 0$, there exists a smooth one-parameter family of metrics $\{g(t)\}$ solving

$$\frac{\partial}{\partial t} g(t) = -2\left(\mathrm{Ric}(g(t)) - \frac{1}{n}\mathscr{E}(g(t))g(t)\right), \quad g(0) = g, \tag{5.7}$$

where $\mathscr{E}$ is the total scalar curvature functional. Moreover, he showed that $g(t)$ converges to a smooth Einstein metric as $t \to \infty$. On a homogeneous manifold, the scalar curvature is constant; thus, $\mathscr{E}(g) = S(g)$. Equation (5.7) reduces to the normalised Ricci flow:

$$\frac{\partial}{\partial t} g(t) = -2\left(\mathrm{Ric}(g(t)) - \frac{1}{n} S(g(t)) g(t)\right). \tag{5.8}$$

In this section, we show that in the classical setting, the scalar curvature is monotone increasing under both the normalised and unnormalised Ricci flows. We also consider the evolution of the scalar curvature in the super setting, providing an explicit example where it is not monotonic.

PROPOSITION 5.6. *Let $M = G/H$ be a homogeneous manifold and $\mathscr{M}_1^G$ be the space of G-invariant, volume one metrics on M. The total scalar curvature functional $\mathscr{E} : \mathscr{M}_1^G \to \mathbb{R}$ is given by $g \mapsto S(g)$ and is monotone increasing under both* (5.8) *and* (1.1).

PROOF. Let $M = G/H$ and let $g \in \mathscr{M}_1^G$. Firstly, because $\mathrm{dvol}_g = 1$, the total scalar curvature functional is given by $\mathscr{E}(g) = S(g)$. Recall the first variation of $\mathscr{E}(g)$:

$$\mathrm{d}\mathscr{E}_g(h) = \int_M \left\langle \frac{S}{2}g - \mathrm{Ric}\, g, h \right\rangle \mathrm{dvol}_g = \left\langle \frac{S}{2}g - \mathrm{Ric}\, g, h \right\rangle,$$

for some $(0,2)$-tensor h. Setting $h = -2\,\mathrm{Ric}\, g + \frac{2}{n}Sg$, one finds the evolution of $\mathscr{E}$ under (5.8) is given by

$$\begin{aligned}\frac{\partial}{\partial t}\mathscr{E}(g) &= \left\langle \frac{S}{2}g, -2\,\mathrm{Ric}\, g \right\rangle + \left\langle \frac{S(g)}{2}g, \frac{2}{n}S(g)g \right\rangle + \langle -\mathrm{Ric}\, g(t), -2\,\mathrm{Ric}\, g\rangle + \left\langle -\mathrm{Ric}\, g, \frac{2}{n}S(g)g \right\rangle \\ &= -S(g)^2 + S(g)^2 + 2|\mathrm{Ric}\, g|^2 - \frac{2}{n}S(g)^2 = 2\left(|\mathrm{Ric}\, g|^2 - \frac{1}{n}S(g)^2\right).\end{aligned}$$

Fixing a basis $\{e_i\}_{i=1}^n$ for $\mathfrak{m}$, we can rewrite this as

$$\begin{aligned}\frac{\partial}{\partial t}\mathscr{E}(g(t)) &= 2\left(\mathrm{tr}\left(\mathrm{Ric}^2 g(t)\right) - \frac{1}{n}\left(\mathrm{tr}\,\mathrm{Ric}\, g(t)\right)^2\right) \\ &= 2\left(\left(\sum_{i=1}^n g(\mathrm{ric}(e_i), e_i)^2\right)\left(\sum_{i=1}^n \left(\frac{1}{\sqrt{n}}\right)^2\right) - \left(\sum_{i=1}^n \frac{1}{\sqrt{n}} g(\mathrm{ric}(e_i), e_i)\right)^2\right) \geq 0,\end{aligned}$$

the inequality holding via Cauchy-Schwarz. With the same proof, we can show the evolution of $\mathscr{E}$ under (1.1) is given by

$$\frac{\partial}{\partial t}\mathscr{E}(g(t)) = 2|\mathrm{Ric}\, g(t)|^2 \geq 0.$$

□

One would hope that a similar result holds for homogeneous superspaces. As we will see, however, the monotonicity does not hold in general. Regardless, the classical theory motivates the following proposition.

PROPOSITION 5.7. *Let $\mathscr{M} = G/H$ be a homogeneous superspace equipped with a G-invariant metric g such that we have the decomposition* (5.3). *Suppose that $g(t)$ is a solution to the normalised Ricci flow* (5.8), *$d_i := \mathrm{sdim}\,\mathfrak{m}_i \neq 0$, and $\sum_{=1}^s d_i \neq 0$. Then, the evolution of the scalar curvature under the* (5.8) *is given by*

$$\frac{\partial}{\partial t}S(g(t)) = 2\left(|\mathrm{Ric}\, g(t)|^2 - \frac{1}{\sum_{i=1}^s d_i}S(g(t))^2\right).$$

PROOF. Fix a Q-normalised basis of each $\mathfrak{m}_i$ such that we have (5.4). Recall that $S(g) = \sum_{i=1}^s r_i d_i$, where r_i is given by (5.5). The normalised Ricci flow equation is given by

$$\frac{\partial}{\partial t}x_i = -2r_i x_i + \frac{2}{\sum_{j=1}^s d_j}\sum_{j=1}^s r_j d_j x_i. \tag{5.9}$$

We naïvely take the time derivative of S under (5.9):

$$\frac{\partial}{\partial t}S(g(t)) = \sum_{m=1}^{s}\frac{\partial x_m}{\partial t}\left(-\frac{b_m d_m}{2x_m^2}+\frac{1}{2}\left(\sum_{\substack{i=1\\i\neq m}}^{s}\left(-\sum_{\substack{k=1\\k\neq m}}^{s}\frac{x_k}{x_m^2 x_i}[imk]+\sum_{\substack{j=1\\j\neq m}}^{s}\frac{1}{x_i x_j}[ijm]\right)\right.\right.$$
$$-\sum_{\substack{j,k=1\\j,k\neq m}}^{s}\frac{x_k}{x_j x_m^2}[mjk]-\sum_{\substack{k=1\\k\neq m}}^{s}\frac{2x_k}{x_m^3}[mmk]-\frac{1}{x_m^2}[mmm]\Bigg)$$
$$\left.\left.-\frac{1}{4}\left(\sum_{\substack{i=1\\i\neq m}}^{s}\left(-\sum_{\substack{k=1\\k\neq m}}^{s}\frac{x_i}{x_k x_m^2}[imk]-\sum_{\substack{j=1\\j\neq m}}^{2}\frac{x_i}{x_j x_m^2}[ijm]-\frac{2x_i}{x_m^3}[imm]\right)+\sum_{\substack{j,k=1\\j,k\neq m}}^{s}\frac{1}{x_j x_k}[mjk]-\frac{1}{x_m^2}[mmm]\right)\right).$$

On the other hand,

$$\begin{aligned}
2\left(|\operatorname{Ric} g|^2-\frac{1}{\sum_{i=1}^{s}d_i}S^2\right) &= 2\left(\operatorname{str}(\operatorname{Ric}^2 g)-\frac{1}{\sum_{i=1}^{s}d_i}(\operatorname{str}\operatorname{Ric} g)^2\right)\\
&= 2\left(\sum_j(-1)^{|e_j|}g(\operatorname{ric}(e_j),\varepsilon_j)^2-\frac{1}{\sum_{i=1}^{s}d_i}\left(\sum_j(-1)^{|e_j|}g(\operatorname{ric}(e_j),\varepsilon_j)\right)^2\right)\\
&= 2\sum_{i=1}^{s}d_i r_i^2-\frac{1}{\sum_{i=1}^{s}d_i}\left(\sum_{i=1}^{s}d_i r_i\right)^2.
\end{aligned}$$

A lengthy computation shows that these two expressions are equal, thus proving the proposition. See Appendix A for a Mathematica script demonstrating this computation. □

REMARK 5.8. *Considering the evolution of $S(g(t))$ under the unnormalised Ricci flow, we find that*

$$\frac{\partial}{\partial t}S(g(t)) = 2|\operatorname{Ric} g|^2.$$

Since $|\cdot|$ depends on the metric g, which can have any signature, this will not be non-negative in general.

Proposition 5.7 characterises the evolution of the scalar curvature under normalised Ricci flow. In contrast to classical theory, we cannot hope to have $\frac{\partial}{\partial t}S \geq 0$ in general. The following example makes this clear.

EXAMPLE 5.9. *Let $G = \mathsf{SU}(3|4)$ and $H = \mathsf{S}(U(2)\times \mathsf{U}(1|2)\times \mathsf{U}(2))$ with Lie superalgebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Consider the homogeneous supermanifold G/H. Set Q to be the supertrace, a non-degenerate even supersymmetric bilinear form, and let g be a G-invariant Riemannian metric on G/H. Fix a Q-orthogonal complement $\mathfrak{m}$ of $\mathfrak{h}$ such that we have the decomposition*

$$\mathfrak{g} = \mathfrak{h}\oplus\mathfrak{m}_1\oplus\mathfrak{m}_2\oplus\mathfrak{m}_3,$$

where $\mathfrak{m}_i \ncong \mathfrak{m}_j$ for $i \neq j$. The associated scalar superproduct on $\mathfrak{m}$ is such that

$$\langle\cdot,\cdot\rangle = x_1 Q|_{\mathfrak{m}_1}\oplus x_2 Q|_{\mathfrak{m}_2}\oplus x_3 Q|_{\mathfrak{m}_3}.$$

It is well known that $B(X,Y) = -2\operatorname{str}(X,Y)$ for $X,Y \in \mathfrak{su}(3|4)$, so $b_i = 2$ for $i = 1,2,3$. Gould, Pulemotov, Rasmussen and Zhang [GPRZ23] *compute $d_1 = -4, d_2 = 4, d_3 = -8$, and the only non-zero structure constant,*

$[123]=-8$. *Hence, the normalised Ricci flow equation becomes*

$$
\begin{aligned}
\frac{\partial}{\partial t}x_1 &= \frac{-3x_1^2+2x_1x_2+x_2^2-(x_1+x_2)x_3+x_3^2}{x_2x_3},\\
\frac{\partial}{\partial t}x_2 &= \frac{-3x_1^2+2x_1x_2+2x_2^2-3x_1x_3+x_2x_3-3x_3^2}{x_1x_3},\\
\frac{\partial}{\partial t}x_3 &= \frac{x_3(x_2-x_1-2x_3)}{x_1x_2}.
\end{aligned}
\tag{5.10}
$$

The evolution of the scalar curvature under (5.10) is given by

$$\frac{\partial}{\partial t}S(g(t)) = \frac{4\left((x_1-x_2)^4+2(x_1-x_2)^2(3x_1+x_2)x_3+(x_1-x_2)(13x_1+7x_2)x_3^2+2(3x_1+x_2)x_3^3+x_3^4\right)}{(x_1x_2x_3)^2}.$$

As $x_1(t), x_2(t)$ and $x_3(t)$ are not sign definite, $\frac{\partial}{\partial t}S(g(t))$ can change sign.

REMARK 5.10. *The same conclusion holds when considering the unnormalised Ricci flow.*

5.3. When G/H is isotropy irreducible

In this section, we prove the following theorem.

THEOREM 5.11. *Let G/H be a homogeneous superspace with irreducible isotropy representation $\mathfrak{m}$. Then, there exists a unique solution*

$$x(t) = \left(\frac{[111]}{2d}-b\right)t + x(0)$$

to the Ricci flow equation on G/H. Furthermore, this solution is defined on a maximal time interval $[0,T)$ and we see one of two possible behaviours occur:

(i) if $\frac{x_0}{C}>0$, where $C:=\frac{[111]}{2d}-b$, then $T<\infty$ is a Type I singularity and G/H shrinks to a point as $t\to T$;
(ii) if $\frac{x_0}{C}<0$, then $T=\infty$ and $x(t)$ diverges to $\pm\infty$.

PROOF. Since $s=1$, we write $x(t), b$, and d to mean $x_1(t), b_1$ and d_1, respectively. Theorem 5.4 implies that the Ricci flow equation on G/H becomes

$$
\begin{aligned}
\frac{\partial}{\partial t}x(t) &= \frac{[111]}{2d}-b,\\
x(0) &= x_0.
\end{aligned}
\tag{5.11}
$$

Integrating (5.11), we find $x(t)=Ct+x_0$, where $C:=\frac{[111]}{2d}-b$. We now consider two cases:

Case 1 ($C>0$): Here $x(t)$ is increasing, so given $x_0<0$, we find $x\left(\frac{x_0}{C}\right)=0$. If $x_0>0$, then $x(t)\to\infty$ as $t\to\infty$.

Case 2 ($C<0$): Here $x(t)$ is decreasing, so given $x_0>0$, we find $x\left(\frac{x_0}{C}\right)=0$. If $x_0<0$, then $x(t)\to-\infty$ as $t\to\infty$.

When $\frac{x_0}{C}>0$, G/H shrinks to a point since $x(t)\to 0$ as $t\to T:=\frac{x_0}{C}$. In this case, $|\mathrm{Rm}(g(t))|^2_{g(t)} = \frac{1}{x(t)^2}|\mathrm{Rm}(Q)|^2_Q$, and so $|\mathrm{Rm}(g(t)|^2_{g(t)}$ blows up like $(T-t)^{-2}$. Thus, $|\mathrm{Rm}(g(t))|_{g(t)}$ blows up like $(T-t)^{-1}$, meaning T is a Type I singularity. □

REMARK 5.12. *Schur's lemma implies that there is a unique left-invariant metric on G/H, which is necessarily Einstein. Rescaling $x(t)$ by $(T-t)^{-1}$, we obtain a metric homothetic to this Einstein metric.*

5.4. When H is not maximal in G

Let G/H be a homogeneous superspace with two irreducible, inequivalent isotropy summands. Assume there exists a subgroup K such that $H < K < G$. Notably, K/H is isotropy irreducible and every G-invariant metric g on G/H is a submersion metric

$$K/H \longrightarrow G/H \longrightarrow G/K$$

by rescaling the metric on the fibre and base. Without loss of generality we define $\mathrm{Lie}(K) = \mathfrak{k} := \mathfrak{h} \oplus \mathfrak{m}_1$. As $[\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{k}$ is Q-orthogonal to $\mathfrak{m}_2$, the structure constants $[112] = [211] = [121]$ vanish. We consider the one parameter family of homogeneous Riemannian metrics

$$g(t) = x_1(t)Q|_{\mathfrak{m}_1} \oplus x_2(t)Q|_{\mathfrak{m}_2},$$

where $x_1(t), x_2(t)$ are real valued smooth functions in t. Proposition 5.2 allows us to compute the Ricci flow equations on G/H:

$$\begin{aligned}\frac{\partial}{\partial t}x_1(t) &= -b_1 + \frac{[111]}{2d_1} + \frac{[122]}{d_1} - \frac{[122]}{2d_1}\left(\frac{x_1(t)}{x_2(t)}\right)^2, \\ \frac{\partial}{\partial t}x_2(t) &= -b_2 + \frac{[222]}{2d_2} + \frac{[122]}{d_2}\frac{x_1(t)}{x_2(t)},\end{aligned} \tag{5.12}$$

where $d_i := \operatorname{sdim}\mathfrak{m}_i$.

REMARK 5.13. *It is possible for the superdimension d_i to be 0, in which case the above formula makes no sense. In this project, we only consider decompositions (5.2) where $d_i \neq 0$ for all $1 \leq i \leq s$.*

As the right hand side of this system and it first derivatives are continuous on the set

$$\mathscr{D} = \{(x_1, x_2) \in \mathbb{R}^2 | x_2 \neq 0\},$$

the Picard-Lindelöff theorem implies that, given initial data $(x_1(0), x_2(0)) \in \mathscr{D}$, there exists a unique solution $(x_1(t), x_2(t))$ defined on any interval I containing $t = 0$. Moreover, $(x_1(t), x_2(t)) \in \mathscr{D}$ for all $t \in I$. By defining

$$A = \frac{[122]}{2d_1}, \quad B = \frac{[122]}{d_2}, \quad C = b_1 - \frac{[111]}{2d_1} - \frac{[122]}{d_1}, \quad D = b_2 - \frac{[222]}{2d_2},$$

(5.12) becomes

$$\begin{aligned}\frac{\partial}{\partial t}x_1(t) &= -C - A\left(\frac{x_1(t)}{x_2(t)}\right)^2, \\ \frac{\partial}{\partial t}x_2(t) &= -D + B\frac{x_1(t)}{x_2(t)}.\end{aligned} \tag{5.13}$$

REMARK 5.14. *The analysis of system (5.13) in the non-super case was conducted in [Buz14]. It is important to note that the supermanifold structure presents a new challenge as A, B, C, and D are not necessarily positive. As a result, the analysis in [Buz14] cannot be directly applied.*

We notice that $A + B = \frac{[122]}{2d_1} + \frac{[122]}{d_2} = 0$ when $[122] = 0$. In this case, system (5.13) becomes

$$\begin{aligned}\frac{\partial}{\partial t}x_1(t) &= \frac{[111]}{2d_1} - b_1, \\ \frac{\partial}{\partial t}x_2(t) &= \frac{[222]}{2d_2} - b_2.\end{aligned}$$

We can adopt a similar approach as in Section 5.3 to analyse the behaviour of this system. Indeed, given initial conditions $(x_1(0), x_2(0)) \in \mathscr{D}$, we find

$$x_1(t) = \left(\frac{[111]}{2d_1} - b_1\right)t + x_1(0),$$
$$x_2(t) = \left(\frac{[222]}{2d_2} - b_2\right)t + x_2(0). \tag{5.14}$$

The following table describes the different behaivour of (5.12) depending on the sign of $\frac{[iii]}{2d_i} - b_i$ for $i = 1, 2$.

TABLE 1. Behaviour of $x_i(t)$ when $A + B = 0$

Initial Conditions	Behaviour when $\frac{[iii]}{2d_i} - b_i > 0$	Converges in finite time	Behaviour when $\frac{[iii]}{2d_i} - b_i < 0$	Converges in finite time
$x_i(0) > 0$	$x_i \to \infty$	No	$x_i \to 0$	Yes
$x_i(0) < 0$	$x_i \to 0$	Yes	$x_i \to -\infty$	No

In the case $x_i(t) \to 0$ as $t \to T$, it is easy to see that $x_i(t) \sim k_i(T - t)$ for some $k_i \in \mathbb{R} \setminus \{0\}$. Proposition 4.5 implies that T is a Type I singularity. Then, by Theorem 4.8, we have the following result.

PROPOSITION 5.15. *Let G/H be a homogeneous superspace where H is not maximal in G. Suppose that the structure constants $[122] = [221] = [212] = 0$. Then the homogeneous Ricci flow, given by (5.14), exhibits one of three behaviours:*

(i) If the inequality $\frac{-x_i(0)}{\frac{[iii]}{2d_i} - b_i} > 0$ holds for exactly one of $i = 1, 2$, then there exists a Type I singularity T such that $x_i(t) \to 0$ as $t \to T$. Furthermore, G/H converges in the Gromov-Hausdorff sense to G/K.
(ii) If $\frac{-x_i(0)}{\frac{[iii]}{2d_i} - b_i} > 0$ holds for both $i = 1$ and $i = 2$, then there exists a Type I singularity T such that $x_i(t) \to 0$ as $t \to T$. Furthermore, the whole space G/H shrinks to a point.
(iii) If $\frac{-x_i(0)}{\frac{[iii]}{2d_i} - b_i} < 0$ for all $i = 1, 2$, then $x_i(t)$ diverges as $t \to \infty$.

Suppose now that $[122] = [221] = [212] \neq 0$. Consider the quantity $y(t) = \frac{x_1(t)}{x_2(t)}$. The evolution of $y(t)$ under (5.12) is given by

$$\frac{\partial}{\partial t} y(t) = \frac{x_2(t)\frac{\partial}{\partial t}x_1(t) - x_1(t)\frac{\partial}{\partial t}x_2(t)}{x_2^2} = \frac{1}{x_2(t)}\left(-(A+B)y(t)^2 + Dy(t) - C\right). \tag{5.15}$$

The critical points of $C - Dy(t) + (A + B)y(t)^2 = 0$ are given by

$$y(t) = \frac{D \pm \left(D^2 - 4C(A+B)\right)^{\frac{1}{2}}}{2(A+B)}.$$

It is well known (for example, see [BWZ04]) that roots of (5.15) correspond to homogeneous Einstein metrics on G/H. The following is a slight generalisation of [Buz14, Lemma 3.3].

LEMMA 5.16. *The quantity $y(t) = \frac{x_1(t)}{x_2(t)}$ is monotone under the homogeneous super Ricci flow.*

PROOF. We consider the three scenarios based on the number of roots of (5.15):

(i) There are no roots of (5.15) and

$$\frac{\partial}{\partial t}y(t) = \frac{1}{x_2}\left(-(A+B)y(t)^2 - Dy(t) + C\right)$$

is sign definite. Given $x_2 > 0$, the conditions $A+B > 0$ and $A+B < 0$ correspond to $\frac{\partial}{\partial t}y < 0$ and $\frac{\partial}{\partial t}y(t) > 0$, respectively. The opposite is true when $x_2 < 0$.

(ii) There is exactly one root, y_1, of (5.15) and

$$\frac{\partial}{\partial t}y(t) = \frac{1}{x_2}\left(-(A+B)y(t)^2 - Dy(t) + C\right) = -\frac{1}{x_2}(A+B)(y-y_1)^2.$$

We find that, for $x_2 > 0$, $\frac{\partial}{\partial t}y(t)$ is positive if $A+B < 0$, and negative if $A+B > 0$. The opposite assertion holds given $x_2 < 0$.

(iii) There are two roots, y_1 and y_2, of (5.15) and

$$\frac{\partial}{\partial t}y(t) = \frac{1}{x_2}\left(-(A+B)y(t)^2 - Dy(t) + C\right) = -\frac{1}{x_2}(A+B)(y-y_1)(y-y_2).$$

Fix $x_2 > 0$ (the case when $x_2 < 0$ is analogous, and so we omit it). Assume, without loss of generality, that $y_1 < y_2$. By the existence and uniqueness of solutions, one cannot flow through y_1 or y_2. Hence, $(y-y_1)(y-y_2)$ does not change sign. The constants A, B, and $y(0)$ determine this sign.

□

Define a reparametrisation of time τ such that $\frac{\mathrm{d}\tau}{\mathrm{d}t} := \frac{1}{x_2(t)}$. By the chain rule,

$$\frac{\mathrm{d}\nu}{\mathrm{d}\tau} = -(A+B)\nu(\tau)^2 + D\nu(\tau) - C, \tag{5.16}$$

where $\nu(\tau) := y(t(\tau))$.

We distinguish three cases based on the number of roots of (5.15):

(I) equation (5.16) has no solutions,

(II) equation (5.16) has one solution $\overline{\nu}$, and

(III) equation (5.16) has two solutions ν_1 and ν_2.

5.4.1. Case (I). By Lemma 5.16, we consider the cases when $A+B > 0$ and $A+B < 0$ separately. Suppose that $A+B > 0$. As (5.16) has no solutions, $\frac{\partial}{\partial \tau}\nu(\tau) < 0$ for all τ and $D^2 < 4C(A+B)$. Let $\nu(0) > 0$. Since $\frac{\partial}{\partial \tau}\nu(\tau)$ does not cross the axis, it is bounded by the turning point:

$$\frac{\partial}{\partial \tau}\nu(\tau) \leq -C - \left(\frac{D^2}{4(A+B)}\right) < 0.$$

Hence, there exists some positive $T \leq \frac{4\nu(0)(A+B)}{D^2 - 4(A+B)C}$ such that $\nu(T) = 0$. If $\nu(0) < 0$, we find that, as $\tau \to \infty$, $\nu(\tau) \to -\infty$. With the same bounds, the opposite assertion holds when $A+B < 0$. The following table describes the possible behaviour when there are no solutions to (5.16).

TABLE 2. Behaviour of $\nu(\tau)$ when (5.16) has no solutions

Initial Conditions	Behaviour when $A+B>0$	Converges in finite time	Behaviour when $A+B<0$	Converges in finite time
$\nu(0)>0$	$\nu\to 0$	Yes	$\nu\to\infty$	No
$\nu(0)<0$	$\nu\to-\infty$	No	$\nu\to 0$	Yes

5.4.2. Case (II)**.** We again consider two cases based on the sign of $A+B$. If $A+B>0$, $\frac{\partial}{\partial\tau}\nu(\tau)<0$ (unless $\nu(0)=\overline{\nu}$, in which case $\frac{\partial}{\partial\tau}\nu(\tau)=0$). We consider separately when the critical point $\overline{\nu}$ is positive, negative, and zero.

Firstly, if $\nu(0)<0$ then $\nu(\tau)$ diverges to $-\infty$ as $\tau\to\infty$. Similarly, if $\nu(0)>\overline{\nu}$ then $\nu(\tau)\to\overline{\nu}$ as $\tau\to\infty$. Suppose that $0<\nu(0)<\overline{\nu}$. We have the estimate

$$\frac{\partial}{\partial\tau}\nu(\tau)\geq-(A+B)\nu(0)^2+D\nu(0)-C.$$

Integrating yields $\nu(\tau)\geq\left(-(A+B)\nu(0)^2+D\nu(0)-C\right)\tau+\nu(0)$, and so there exists some $T\leq\frac{\nu(0)}{-(A+B)\nu(0)^2+D\nu(0)-C}$ such that $\nu(T)=0$.

If $\overline{\nu}<0$, we see similar behaviour: suppose $\overline{\nu}<\nu(0)<0$, then $\nu(\tau)\to\overline{\nu}$ as $\tau\to\infty$. Given $\nu(0)<\overline{\nu}$, then $\nu(\tau)\to-\infty$ as $\tau\to\infty$. If $\nu(0)>0$, we can bound $\frac{\partial}{\partial\tau}\nu(\tau)\leq-C$ for all τ such that $\frac{\partial}{\partial\tau}\nu(\tau)>0$. Integrating this we find $\nu(\tau)\leq-C\tau+\nu(0)$, and so there exists some $T\leq\frac{\nu(0)}{C}$ such that $\nu(T)=0$.

Finally, if $\overline{\nu}=0$, we see $\nu(\tau)$ either converge to 0 or diverge to $-\infty$ as $\tau\to\infty$. We remark that this occurs if and only if $C=0$. The analysis when $A+B<0$ is very similar, and so we omit it. The following table represents the possible behaviours of $\nu(\tau)$.

TABLE 3. Behaviour of $\nu(\tau)$ when (5.16) has one solution

Initial Conditions	Behaviour when $A+B>0$	Converges in finite time	Behaviour when $A+B<0$	Converges in finite time
$0<\overline{\nu}<\nu(0)$	$\nu\to\overline{\nu}$	No	$\nu\to\infty$	No
$0<\nu(0)<\overline{\nu}$	$\nu\to 0$	Yes	$\nu\to\overline{\nu}$	Yes
$\nu(0)<0<\overline{\nu}$	$\nu\to-\infty$	No	$\nu\to 0$	No
$\overline{\nu}<0<\nu(0)$	$\nu\to 0$	No	$\nu\to\infty$	No
$\overline{\nu}<\nu(0)<0$	$\nu\to\overline{\nu}$	No	$\nu\to 0$	Yes
$\nu(0)<\overline{\nu}<0$	$\nu\to-\infty$	No	$\nu\to\overline{\nu}$	No
$\overline{\nu}=0<\nu(0)$	$y\to-\infty$	No	$\nu\to 0$	No
$\overline{\nu}=0>\nu(0)$	$\nu\to 0$	No	$\nu\to\infty$	No

5.4.3. Case (III)**.** In this case we have two solutions ν_1 and ν_2. Assuming, without loss of generality, that $\nu_1<\nu_2$, there are five possible ways to arrange the two solutions about 0. Using the same two bounds on $\frac{\partial}{\partial\tau}\nu(\tau)$ (either $\frac{\partial}{\partial\tau}\nu(\tau)\geq-(A+B)\nu(0)^2+D\nu(0)-C$ or $\frac{\partial}{\partial\tau}\nu(\tau)\leq-C$), one can show that in certain cases there exists $T<\infty$ such that $\nu(T)=0$. The following table describes the behaviour of $\nu(\tau)$ under the assumptions of (III).

TABLE 4. Behaviour of $\nu(\tau)$ when (5.16) has two solutions

Initial Conditions	Behaviour when $A+B>0$	Converges in finite time	Behaviour when $A+B<0$	Converges in finite time
$0<\nu_1<\nu_2<\nu(0)$	$\nu\to\nu_2$	No	$\nu\to\infty$	No
$0<\nu_1<\nu(0)<\nu_2$	$\nu\to\nu_2$	No	$\nu\to\nu_1$	No
$0<\nu(0)<\nu_1<\nu_2$	$\nu\to 0$	Yes	$\nu\to\nu_1$	No
$\nu(0)<0<\nu_1<\nu_2$	$\nu\to-\infty$	No	$\nu\to 0$	Yes
$\nu_1<0<\nu_2<\nu(0)$	$\nu\to\nu_2$	No	$\nu\to\infty$	No
$\nu_1<0<\nu(0)<\nu_2$	$\nu\to\nu_2$	No	$\nu\to 0$	Yes
$\nu_1<\nu(0)<0<\nu_2$	$\nu\to 0$	Yes	$\nu\to\nu_1$	No
$\nu(0)<\nu_1<0<\nu_2$	$\nu\to-\infty$	No	$\nu\to\nu_1$	No
$\nu_1<\nu_2<0<\nu(0)$	$\nu\to 0$	Yes	$\nu\to\infty$	No
$\nu_1<\nu_2<\nu(0)<0$	$\nu\to\nu_2$	No	$\nu\to 0$	Yes
$\nu_1<\nu(0)<\nu_2<0$	$\nu\to\nu_2$	No	$\nu\to\nu_1$	No
$\nu(0)<\nu_1<\nu_2<0$	$\nu\to-\infty$	No	$\nu\to\nu_1$	No
$0<\nu_2<\nu(0)$	$\nu\to\nu_2$	No	$\nu\to\infty$	No
$0<\nu(0)<\nu_2$	$\nu\to\nu_2$	No	$\nu\to 0$	No
$\nu(0)<0<\nu_2$	$\nu\to-\infty$	No	$\nu\to 0$	No
$\nu_1<0<\nu(0)$	$\nu\to 0$	No	$\nu\to\infty$	No
$\nu_1<\nu(0)<0$	$\nu\to\nu_1$	No	$\nu\to 0$	No
$\nu(0)<\nu_1<0$	$\nu\to-\infty$	No	$\nu\to\nu_1$	No

Unfortunately we were unable to investigate the asymptotics in the original time variable. Undoing the reparametrisation $t\rightsquigarrow\tau$ appears to be quite subtle. However, we have the following.

CONJECTURE A. *Let G/H be a homogeneous superspace where H is not maximal in G and consider a homogeneous G-invariant Riemannian metric of the form* (1.3)*. If $(x_1(0),x_2(0))\in\{(x_1,x_2)\in\mathbb{R}^2 : x_1\neq 0, x_2>0\}$, then the behaviour of the quantity $\frac{x_1(t)}{x_2(t)}$ under the homogeneous super Ricci flow is as in Tables 2, 3, and 4.*

On the other hand, if $x_2(0)<0$ and $A+B>0$ (resp. $A+B<0$), then the behaviour of $\frac{x_1(t)}{x_2(t)}$ under the homogeneous super Ricci flow is as in Tables 2, 3, and 4, found on pages 49 *and* 50*, corresponding to the columns with $A+B<0$ (resp. $A+B>0$).*

We suspect that the different behaviour when $x_2(0)<0$ occurs because of the presence of $x_2(t)$ in the reparametrisation $\tau(t)=\int_0^t \frac{1}{x_2(s)}\mathrm{d}s$.

Conjecture *A*, if proven true, implies that the Ricci flow of homogeneous superspaces with two isotropy summands has different behaviour, in general, to its non-super counterpart. We now give an explicit example of an infinite family of homogeneous supermanifolds which do not exhibit a finite time singularity. This example, and many others can be constructed similarly to the homogeneous spaces in [DK08, Section 5.2].

5.5. An example without finite time singularities

To construct the following example, we first consider an analogous space in the non-super setting. To show that the isotropy representation of this homogeneous space splits into two inequivalent irreducible summands, we first discuss some general theory of generalised flag manifolds, following the exposition in [AC09].

5.5.1. Generalised flag manifolds and their isotropy representations. Let G be a compact semisimple Lie group. A compact connected abelian Lie subgroup of G is called a *torus*. A *generalised flag manifold* is a homogeneous space G/K where the isotropy group K is the centraliser of a torus in G. Consider a generalised flag manifold G/K. Let $\mathfrak{g}^{\mathbb{C}}$ denote the complexified Lie algebra of G and $\mathfrak{h}^{\mathbb{C}}$ denote the Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. With the notation of section 2.2, we fix a Cartan-Weyl basis $\{H_\alpha, E_\alpha\}$ of $\mathfrak{g}^{\mathbb{C}}$ with compact real form

$$\mathfrak{g} = \sqrt{-1}\mathfrak{h}_{\mathbb{R}} \oplus \bigoplus_{\alpha\in\Delta^+} \mathbb{R}A_\alpha \oplus \bigoplus_{\alpha\in\Delta^+} \mathbb{R}\sqrt{-1}B_\alpha.$$

Let Π_K be a subset of $\Pi = \{\alpha_1, \dots, \alpha_l\}$, where $l = \dim \mathfrak{h}^{\mathbb{C}}$, and define $\Pi_M := \Pi \setminus \Pi_K$. Denote by Δ_K the closed subsystem spanned by Π_K, and set $\Delta_M = \Delta \setminus \Delta_K$. We have the following subsystems of Δ:

$$\Delta_K^+ := \Delta^+ \cap \langle \Pi_K \rangle, \quad \Delta_M^+ := \Delta^+ \cap \Delta_M,$$
$$\Delta_K^- := \Delta^- \cap \langle \Pi_K \rangle, \quad \Delta_M^- := \Delta^- \cap \Delta_M,$$

where $\langle \Pi_K \rangle$ denotes the roots generated by Π_K. The set

$$\mathfrak{k} = \sqrt{-1}\mathfrak{h}_{\mathbb{R}} \oplus \bigoplus_{\alpha\in\Delta_K^+} \mathbb{R}A_\alpha \oplus \bigoplus_{\alpha\in\Delta_K^+} \mathbb{R}\sqrt{-1}B_\alpha.$$

is a real subalgebra of $\mathfrak{g}$ corresponding to the Lie group K [AC09]. We define a linear subspace of $\mathfrak{g}$ by

$$\mathfrak{m} := \bigoplus_{\alpha\in\Delta_M^+} \mathbb{R}A_\alpha \oplus \bigoplus_{\alpha\in\Delta_M^+} \mathbb{R}\sqrt{-1}B_\alpha.$$

We thus have a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$. We identify the space $\mathfrak{m}$ with the tangent space $T_K(G/K)$. As usual, the isotropy representation of K on $T_K(G/K)$ is identified with the restriction of the adjoint representation of K on $\mathfrak{m}$, $\mathrm{Ad}^K|_{\mathfrak{m}}$.

Suppose that $\Pi_M = \{\alpha_{i_1}, \dots, \alpha_{i_r}\}$ for $1 \le i_1, \dots, i_r \le l$. For any positive integers $k_1, \dots, k_r$, define

$$\Delta(k_1, \dots, k_r) = \left\{ \sum_{j=1}^{l} m_j \alpha_j : m_{i_1} = k_1, \dots, m_{i_r} = k_r \right\}.$$

If $\Delta(k_1, \dots, k_r) \neq \emptyset$, we define an Ad^K-invariant subspace of $\mathfrak{g}$ by

$$\mathfrak{m}(k_1, \dots, k_r) := \bigoplus_{\alpha\in\Delta(k_1,\dots,k_r)} \mathbb{R}A_\alpha \oplus \bigoplus_{\alpha\in\Delta(k_1,\dots,k_r)} \mathbb{R}\sqrt{-1}B_\alpha.$$

Then,

$$\mathfrak{m} = \bigoplus_{k_1,\dots,k_r} \mathfrak{m}(k_1, \dots, k_r).$$

Let $\mathfrak{m}^{\pm}(k_1,\dots,k_r) = \sum_{\alpha\in\Delta(k_1,\dots,k_r)} \mathfrak{g}^{\mathbb{C}}_{\pm\alpha}$, then

$$\mathfrak{m}(k_1,\dots,k_r)^{\mathbb{C}} = \mathfrak{m}^{+}(k_1,\dots,k_r) \oplus \mathfrak{m}^{-}(k_1,\dots,k_r)$$

Let $\mathfrak{k}' = [\mathfrak{k}^{\mathbb{C}},\mathfrak{k}^{\mathbb{C}}]$ be the semisimple part of $\mathfrak{k}^{\mathbb{C}}$. We have the following sufficiency condition for the irreducibility of the Ad^K-module $\mathfrak{m}(k_1,\dots,k_r)$.

THEOREM 5.17. [Kim90, Lemma 2.2] *For each $\mathfrak{m}(k_1,\dots,k_r)$, the following are equivalent:*

(i) $\mathrm{ad}_{\mathfrak{g}}|_{\mathfrak{k}} : \mathfrak{k} \to \mathfrak{gl}(\mathfrak{m}(k_1,\dots,k_r))$ *is a real irreducible representation of* $\mathfrak{k}$,
(ii) $\mathrm{ad}_{\mathfrak{g}^{\mathbb{C}}}|_{\mathfrak{k}'} : \mathfrak{k}' \to \mathfrak{gl}(\mathfrak{m}^{+}(k_1,\dots,k_r))$ *is a complex irreducible representation of* $\mathfrak{k}'$,
(iii) $\mathrm{ad}_{\mathfrak{g}^{\mathbb{C}}}|_{\mathfrak{k}'} : \mathfrak{k}' \to \mathfrak{gl}(\mathfrak{m}^{-}(k_1,\dots,k_r))$ *is a complex irreducible representation of* $\mathfrak{k}'$.

5.5.2. An example in the non-super setting. In this section, we show that the homogeneous space $\mathsf{SU}(pq+m)/\mathsf{SU}(p)\times\mathsf{SU}(q)\times\mathsf{U}(m)$ has two inequivalent irreducible isotropy summands. Wolf [Wol68] gives an irreducible unitary representation of $\mathsf{SU}(p)\times\mathsf{SU}(q)$

$$\pi : \mathsf{SU}(p)\times\mathsf{SU}(q) \to \mathsf{SU}(pq),$$

where the inclusion is the tensor product of the natural representations of $\mathsf{SU}(p)$ and $\mathsf{SU}(q)$. In other words,

$$d\pi^{\mathbb{C}} : \mathfrak{sl}(\mathbb{C}^p)\oplus\mathfrak{sl}(\mathbb{C}^q) \to \mathfrak{sl}(\mathbb{C}^p\otimes\mathbb{C}^q)$$

is irreducible. We extend this inclusion to

$$\widetilde{\pi} : \mathsf{SU}(p)\times\mathsf{SU}(q)\times\mathsf{U}(m) \to \mathsf{SU}(pq)\times\mathsf{U}(m),$$

where $\widetilde{\pi}(g,h) = (\pi(g),h)$. With this in mind, consider the Lie groups $H \le K \le G$, where

$$H := \mathsf{SU}(p)\times\mathsf{SU}(q)\times\mathsf{U}(m),\ K := \mathsf{SU}(pq)\times\mathsf{U}(m),\ \text{and } G := \mathsf{SU}(pq+m).$$

In order to show that G/H has two irreducible inequivalent isotropy summands, we must show three facts:

- G/K is isotropy irreducible;
- the isotropy representation of K on $T_K(G/K)$ restricted to H is irreducible;
- K/H is isotropy irreducible.

Denote by $\mathfrak{h},\mathfrak{k}$, and $\mathfrak{g}$, the Lie algebras of H,K, and G, respectively. Let us first show that G/K is isotropy irreducible. Consider the complexified Lie algebras $\mathfrak{k}^{\mathbb{C}} = \mathfrak{sl}(\mathbb{C}^p\otimes\mathbb{C}^q)\oplus\mathfrak{gl}(\mathbb{C}^m)$ and $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}(\mathbb{C}^p\otimes\mathbb{C}^q\oplus\mathbb{C}^m)$ of $\mathfrak{k}$ and $\mathfrak{g}$. The root systems of G and K are

$$\Delta = \{\varepsilon_i-\varepsilon_j : 1\le i\ne j\le pq+m\},$$
$$\Delta_K = \{\varepsilon_i-\varepsilon_j : 1\le i\ne j\le pq\}\cup\{\varepsilon_i-\varepsilon_j : pq+1\le i\ne j\le pq+m\}.$$

Hence,

$$\Delta_M = \{\varepsilon_i-\varepsilon_j, \varepsilon_j-\varepsilon_i : 1\le i\le pq,\ pq+1\le j\le pq+m\}$$

is the set of roots corresponding to $\mathfrak{m}^{\mathbb{C}}$, which we identify with the complexified tangent space $T_K(G/K)^{\mathbb{C}} \cong \mathfrak{g}^{\mathbb{C}}/\mathfrak{k}^{\mathbb{C}}$. Define $\Delta^{\pm}_M := \Delta_M\cap\Delta^{\pm}$. Notice that the Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$ is the traceless diagonal matrices of size $pq+m$, and so $|\Pi| = pq+m-1$. Hence, $\Pi_M = \Pi\setminus\Pi_K = \{\alpha_{i_1}\}$ for some $1\le i_1\le pq+m-1$. Define

$$\Delta(k) = \left\{\sum_{i=1}^{pq+m-1} m_i\alpha_i : m_{i_1} = k\right\}$$

for each positive integer k. Since the coefficients of positive roots of Type A are either 1 or 0, $\Delta(k) \neq \emptyset$ requires $k = 1$. Hence, $\Delta(1) = \Delta_M^+$, allowing us to define

$$\mathfrak{m}_{\pm 2}^{\mathbb{C}} := \mathfrak{m}^{\pm}(1) = \bigoplus_{\alpha \in \Delta_M^+} \mathfrak{g}_{\pm\alpha}^{\mathbb{C}}.$$

Explicitly,

$$\mathfrak{m}_{+2}^{\mathbb{C}} = \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} : B \in \mathsf{M}_{pq,m}(\mathbb{C}) \right\}, \text{ and } \mathfrak{m}_{-2}^{\mathbb{C}} = \left\{ \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} : C \in \mathsf{M}_{m,pq}(\mathbb{C}) \right\}.$$

We now compute the adjoint representation of $\mathfrak{k}^{\mathbb{C}}$ on $\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}_2^{\mathbb{C}} \oplus \mathfrak{m}_{-2}^{\mathbb{C}}$. Indeed, let $X \in \mathfrak{sl}(\mathbb{C}^p \otimes \mathbb{C}^q) \oplus \mathfrak{gl}(\mathbb{C}^m)$ have block form

$$X = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}, \quad X_1 \in \mathfrak{sl}(\mathbb{C}^p \otimes \mathbb{C}^q), X_2 \in \mathfrak{gl}(\mathbb{C}^m).$$

We know $Y \in \mathfrak{m}$ has block form

$$Y = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}, \text{ where } B \in \mathsf{M}_{pq,m}(\mathbb{C}), C \in \mathsf{M}_{m,pq}(\mathbb{C}),$$

so consider

$$Y_1 = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \in \mathfrak{m}_2^{\mathbb{C}} \text{ and } Y_2 = \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \in \mathfrak{m}_{-2}^{\mathbb{C}}.$$

Then,

$$\mathrm{ad}_X(Y_1) = \begin{pmatrix} 0 & X_1 B - B X_2 \\ 0 & 0 \end{pmatrix}, \quad \mathrm{ad}_X(Y_2) = \begin{pmatrix} 0 & 0 \\ X_2 C - C X_1 & 0 \end{pmatrix}.$$

Hence, $\mathfrak{m}_{+2}^{\mathbb{C}} \cong \mathbb{C}^p \otimes \mathbb{C}^q \otimes (\mathbb{C}^m)^*$ and $\mathfrak{m}_{-2}^{\mathbb{C}} \cong (\mathbb{C}^p \otimes \mathbb{C}^q)^* \otimes \mathbb{C}^m$. The action of $\mathfrak{k}^{\mathbb{C}} \cong \mathfrak{sl}(\mathbb{C}^p \otimes \mathbb{C}^q) \oplus \mathfrak{gl}(\mathbb{C}^m)$ on $\mathfrak{m}_{+2}^{\mathbb{C}} \cong \mathbb{C}^p \otimes \mathbb{C}^q \otimes (\mathbb{C}^m)^*$ is given by the tensor product of the natural representation of $\mathfrak{sl}(\mathbb{C}^p \otimes \mathbb{C}^q)$ and the dual of the natural representation of $\mathfrak{gl}(\mathbb{C}^m)$. Similarly, the action of $\mathfrak{k}^{\mathbb{C}} \cong \mathfrak{sl}(\mathbb{C}^p \otimes \mathbb{C}^q) \oplus \mathfrak{gl}(\mathbb{C}^m)$ on $\mathfrak{m}_{-2}^{\mathbb{C}} \cong (\mathbb{C}^p \otimes \mathbb{C}^q)^* \otimes \mathbb{C}^m$ is given by tensor product of the dual of the natural representation of $\mathfrak{sl}(\mathbb{C}^p \otimes \mathbb{C}^q)$ and the natural representation of $\mathfrak{gl}(\mathbb{C}^m)$. Since the tensor product of irreducible representations is irreducible, $\mathfrak{m}_{\pm 2}^{\mathbb{C}}$ are irreducible $\mathrm{ad}_{\mathfrak{k}}$-representations. By Theorem 5.17, $\mathfrak{m}$ is $\mathrm{ad}_{\mathfrak{k}}$-irreducible. Since K is connected and $\mathfrak{m}$ is $\mathrm{ad}_{\mathfrak{k}}$-irreducible, the adjoint representation of K on $T_K(G/K) \cong \mathfrak{g}/\mathfrak{k}$ is irreducible. This is equivalent to the isotropy representation of K on $T_K(G/K)$ being irreducible.

It follows that the restriction of the isotropy rep of K on $T_K(G/K)$ to H remains irreducible. Indeed, the action of $\mathfrak{k}^{\mathbb{C}}$ on $\mathfrak{m}_{+2}^{\mathbb{C}}$ restricted to $\mathfrak{h}^{\mathbb{C}}$ is the tensor product of natural representations of $\mathfrak{sl}(\mathbb{C}^p)$ and $\mathfrak{sl}(\mathbb{C}^q)$, and the dual of the natural representation of $\mathfrak{gl}(\mathbb{C}^m)$. The restriction of the action of $\mathfrak{k}^{\mathbb{C}}$ on $\mathfrak{m}_{-2}^{\mathbb{C}}$ is similarly irreducible.

We finally show that K/H is isotropy irreducible. Wolf [Wol68] shows that the isotropy representation of $\mathsf{SU}(pq)/\mathsf{SU}(p) \times \mathsf{SU}(q)$ is the tensor product of the adjoint representations of $\mathsf{SU}(p)$ and $\mathsf{SU}(q)$, which is irreducible with dimension $(p^2-1)(q^2-1)$. The action of $\mathfrak{h}^{\mathbb{C}} \cong \mathfrak{sl}(\mathbb{C}^p) \oplus \mathfrak{sl}(\mathbb{C}^q) \oplus \mathfrak{gl}(\mathbb{C}^m)$ on the complexified tangent space $\mathfrak{k}^{\mathbb{C}}/\mathfrak{h}^{\mathbb{C}} \cong \mathfrak{sl}(\mathbb{C}^p) \otimes \mathfrak{sl}(\mathbb{C}^q)$ is thus given by the tensor product of the adjoint representations of $\mathfrak{sl}(\mathbb{C}^p)$ and $\mathfrak{sl}(\mathbb{C}^q)$, and the trivial representation of $\mathfrak{gl}(\mathbb{C}^m)$. Hence, the isotropy representation $\mathfrak{m}_1$ of K/H is irreducible with dimension $(p^2-1)(q^2-1)$.

We now verify that the sum of the dimensions of $\mathfrak{m}_1$ and $\mathfrak{m}_2$ match the dimension of the tangent space $T_H(G/H)$. This shows that $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$. Indeed,

$$\dim \mathfrak{m}_1 = (p^2-1)(q^2-1),$$
$$\dim \mathfrak{m}_2 = 2pqm,$$

$$\dim \mathfrak{g}/\mathfrak{h} = (pq+m)^2 - 1 - (p^2 - 1 + q^2 - 1 + m^2) = p^2q^2 + 2pqm - p^2 - q^2 + 1 = \dim \mathfrak{m}_1 + \dim \mathfrak{m}_2.$$

5.5.3. An analogue in the super setting. Consider the same inclusion [Wol68]: $\pi : \mathsf{SU}(p) \times \mathsf{SU}(q) \to \mathsf{SU}(pq)$. This induces an inclusion of Lie supergroups $\widetilde{\pi} : \mathsf{SU}(p) \times \mathsf{SU}(q) \times \mathsf{U}(m|n) \to \mathsf{SU}(pq) \times \mathsf{U}(m|n)$. With this in mind, we define the Lie supergroups $H \le K \le G$:

$$H = \mathsf{SU}(p) \times \mathsf{SU}(q) \times \mathsf{U}(m|n),\ K = \mathsf{SU}(pq) \times \mathsf{U}(m|n),\ \text{and } G = \mathsf{SU}(pq+m|n).$$

According to [Kac77], the root system of G and the subsystem corresponding to K are

$$\Delta = \left\{ \varepsilon_i - \varepsilon_j, \delta_k - \delta_l : \begin{array}{l} 1 \le i \ne j \le pq+m, \\ 1 \le k \ne l \le n \end{array} \right\} \cup \left\{ \pm(\varepsilon_i - \delta_j) : \begin{array}{l} 1 \le i \le pq+m \\ 1 \le j \le n \end{array} \right\},$$

$$\Delta_K = \left\{ \varepsilon_i - \varepsilon_j : 1 \le i \ne j \le pq \right\} \cup \left\{ \varepsilon_i - \varepsilon_j, \delta_k - \delta_l : \begin{array}{l} pq+1 \le i \ne j \le pq+m \\ 1 \le k \ne l \le n \end{array} \right\}$$
$$\cup \left\{ \pm(\varepsilon_i - \delta_j) : \begin{array}{l} pq+1 \le i \le pq+m \\ 1 \le j \le n \end{array} \right\}.$$

Hence

$$\Delta_M = \Delta \setminus \Delta_K = \left\{ \pm(\varepsilon_i - \varepsilon_j) : \begin{array}{l} 1 \le i \le pq, \\ pq+1 \le j \le pq+m \end{array} \right\} \cup \left\{ \pm(\varepsilon_i - \delta_j) : \begin{array}{l} 1 \le i \le pq, \\ 1 \le j \le n \end{array} \right\}$$

is the set of roots corresponding to $\mathfrak{m}^{\mathbb{C}}$, which we identify with the complexified tangent space $T_K(G/K)$. As usual, let $\Delta_M^+ = \Delta^+ \cap \Delta_M$ (resp. $\Delta_M^- = \Delta^- \cap \Delta_M$) denote the set of positive (resp. negative) roots in Δ_M. The simple roots in Δ and Δ_K are:

$$\Pi = \left\{ \varepsilon_{i-1} - \varepsilon_i, \varepsilon_{pq+m} - \delta_1, \delta_{j-1} - \delta_j : \begin{array}{l} 2 \le i \le pq+m \\ 2 \le j \le n \end{array} \right\}$$

$$\Pi_K = \{ \varepsilon_{i-1} - \varepsilon_i : 2 \le i \le pq \} \cup \left\{ \varepsilon_{i-1} - \varepsilon_i, \varepsilon_{pq+m} - \delta_1, \delta_{j-1} - \delta_j : \begin{array}{l} pq+2 \le i \le pq+m \\ 2 \le j \le n \end{array} \right\}$$

Hence $\Pi_M = \{\varepsilon_{pq} - \varepsilon_{pq+1}\}$, and so we define

$$\Delta(k) = \left\{ \sum_{i=1}^{pq+m+n-1} m_i \alpha_i : \alpha_i \in \Pi, m_{pq+1} = k \right\}.$$

As in the non-super setting, k must be 1 allowing us to define

$$\mathfrak{m}^{\mathbb{C}}_{\pm 2} := \mathfrak{m}^{\pm}(1) = \bigoplus_{\alpha \in \Delta_M^+} \mathfrak{g}^{\mathbb{C}}_{\pm\alpha}.$$

Hence, $\mathfrak{m}^{\mathbb{C}}_{+2} \cong \mathbb{C}^p \otimes \mathbb{C}^q \otimes (\mathbb{C}^{m|n})^*$, and $\mathfrak{m}^{\mathbb{C}}_{-2} \cong (\mathbb{C}^p \otimes \mathbb{C}^q)^* \otimes \mathbb{C}^{m|n}$. These are both irreducible $\mathrm{ad}_{\mathfrak{k}}$-modules since the action of $\mathfrak{k}^{\mathbb{C}}$ on $\mathfrak{m}^{\mathbb{C}}_{+2}$ (resp. $\mathfrak{m}^{\mathbb{C}}_{-2}$) is given by the tensor product of the natural (resp. dual of natural) representation of $\mathfrak{sl}(\mathbb{C}^p \otimes \mathbb{C}^q)$ and the dual of the natural (resp. natural) representation of $\mathfrak{gl}(\mathbb{C}^{m|n})$. By the same reasoning as the non-super setting, the restriction to $\mathfrak{h}^{\mathbb{C}}$ remains irreducible. Thus, by 5.17, both the isotropy representation of G/K, $\mathfrak{m}_2$, and its restriction to H are irreducible.

Finally, the action of $\mathfrak{h}^{\mathbb{C}} \cong \mathfrak{sl}(\mathbb{C}^p) \oplus \mathfrak{sl}(\mathbb{C}^q) \oplus \mathfrak{gl}(\mathbb{C}^{m|n})$ on the complexified tangent space $\mathfrak{k}^{\mathbb{C}}/\mathfrak{h}^{\mathbb{C}} \cong \mathfrak{sl}(\mathbb{C}^p) \otimes \mathfrak{sl}(\mathbb{C}^q)$ is given by the tensor product of the adjoint representations of $\mathfrak{sl}(\mathbb{C}^p)$ and $\mathfrak{sl}(\mathbb{C}^q)$, and the trivial representation of $\mathfrak{gl}(\mathbb{C}^{m|n})$. Hence, the isotropy representation $\mathfrak{m}_1$ of K/H is irreducible.

It follows that the isotropy representation of G/H splits into two inequivalent irreducible summands $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$, where

$$d_1 := \operatorname{sdim} \mathfrak{m}_1 = (p^2-1)(q^2-1) \ \text{ and } \ d_2 := \operatorname{sdim} \mathfrak{m}_2 = 2pq(m-n).$$

The ratio of the Killing forms of G and K is $\alpha = \frac{pq}{pq+m-n}$, and since $\mathfrak{g}$ is simple, $b_1 = b_2 = -2(pq+m-n)$. We compute

$$[122] = b_1 d_1 (1-\alpha) = -2(p^2-1)(q^2-1)(m-n),$$

and all other structure constants are zero.

The homogeneous Ricci flow equation of a G-invariant metric on $\mathsf{SU}(pq+m|n)/\mathsf{S}(\mathsf{SU}(p)\mathsf{SU}(q)\mathsf{U}(1)\mathsf{U}(m|n))$ becomes (5.13) where

$$A = n-m, \quad B = -\frac{(p^2-1)(q^2-1)}{pq}, \quad C = -2pq, \quad D = -2(pq+m-n).$$

Notice that B and C are always negative. In the non-super setting, [Böh15, Theorem 2] implies that any homogeneous manifold not diffeomorphic to the torus has finite extinction time. We aim to show that in this example, the Ricci flow does not have finite extinction time. Let $\Sigma := \left\{ (x_1, x_2) \in \mathbb{R}^2 \middle| x_1 \geq 0, x_2 > 0 \right\}$, and consider initial conditions $(x_1(0), x_2(0)) \in \Sigma$. Fix $m > n$. In this case, $A < 0$, so $\frac{\partial}{\partial t} x_1(t) > 0$ for all t. Consider the line

$$L := \left\{ (x_1, x_2) \in \Sigma \,\middle|\, x_2 = \frac{B}{D} x_1 \right\}.$$

This line splits Σ into two connected regions:

$$\Sigma_1 := \left\{ (x_1, x_2) \in \Sigma \,\middle|\, x_2 < \frac{B}{D} x_1 \right\}, \text{ and}$$
$$\Sigma_2 := \left\{ (x_1, x_2) \in \Sigma \,\middle|\, x_2 > \frac{B}{D} x_1 \right\}.$$

In Σ_1, $x_2(t)$ is monotone increasing, while in Σ_2, $x_2(t)$ is monotone decreasing.

If $(x_1(t), x_2(t)) \in \Sigma_2$, then there exists $t_1 > t$ such that $(x_1(t_1), x_2(t_1)) \in L$. Since points in L instantly enter Σ_1 ($\frac{\partial}{\partial t} x_2 = 0$ and $\frac{\partial}{\partial t} x_1 > 0$), we conclude that any initial condition in Σ_2 will eventually enter Σ_1. Furthermore, points in Σ_1 have both $x_1(t)$ and $x_2(t)$ increasing. Hence, every solution blows up as $t \to \infty$. It is clear that solutions cannot blow up in finite time.

5.6. An example with finite time singularities

For the remainder of this section, fix $G = \mathsf{SOSp}(2|2n)$ and $H = \mathsf{U}(1|p-1) \times \mathsf{Sp}(2(n+1-p))$. Here $2 \leq p \leq n$ denotes the node removed from the Dynkin diagram of type $C(n+1)$. This is a specific example of the previous situation, where we have an intermediate subgroup $K = \mathsf{SOSp}(2|2(p-1)) \times \mathsf{Sp}(2(n+1-p))$. It can be shown (see [GPRZ23]) that the isotropy representation $\mathfrak{m}$ of G/H splits into two irreducible, inequivalent summands $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ satisfying the following relations:

$$[\mathfrak{m}_2, \mathfrak{m}_2] \subset \mathfrak{h} \oplus \mathfrak{m}_1, \quad [\mathfrak{m}_2, \mathfrak{m}_1] \subset \mathfrak{m}_2, \quad [\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{h}.$$

The superdimension of the modules $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are $d_1 = (p-1)(p-2)$ and $d_2 = 4(p-2)(n+1-p)$ respectively. Let g be a G-invariant metric on G/H. The associated scalar superproduct can be written as

$$\langle \cdot, \cdot \rangle = x_1 Q|_{\mathfrak{m}_1} \oplus x_2\, Q|_{\mathfrak{m}_2} \tag{5.17}$$

for $x_1, x_2 \neq 0$. As we saw in Example 3.8, $-2n\,\mathrm{str}(X,Y) = B(X,Y)$ for all $X, Y \in \mathfrak{g}$, hence taking $Q = \mathrm{str}$ gives $b_i = 2n$ for $i = 1, 2$. The above inclusions imply all structure constants except $[221] = [212] = [122] = 2(n+1-p)(p-1)(p-2)$ are zero. We remark that it is possible to have all structure constants vanish (for example, take $p = 2$).

Substituting our expressions for b_i, d_1, d_2 and $[122]$ into (5.12), the Ricci flow equation on G/H becomes the system of ODEs

$$\begin{aligned} \frac{\partial}{\partial t}x_1(t) &= 2-2p-(n+1-p)\frac{x_1(t)^2}{x_2(t)^2}, \\ \frac{\partial}{\partial t}x_2(t) &= -2n+\frac{(p-1)}{2}\frac{x_1(t)}{x_2(t)}. \end{aligned} \tag{5.18}$$

Since $2 \le p \le n$, we have the estimate

$$\frac{\partial}{\partial t}x_1(t) \le 2-2p < 0 \tag{5.19}$$

for all $x_1(t) \in \mathbb{R}$ and $x_2(t) \in \mathbb{R}\setminus\{0\}$. To analyse (5.18), we consider four cases:

(I) $x_1(0) \ge 0$ and $x_2(0) > 0$,

(II) $x_1(0) \ge 0$ and $x_2(0) < 0$,

(III) $x_1(0) < 0$ and $x_2(0) > 0$, and

(IV) $x_1(0) < 0$ and $x_2(0) < 0$.

Case I. This is a specific case of the dynamical system analysed in [Buz14, Theorem 3.4]. We see this by defining the variables

$$A = n+1-p, \quad B = \frac{p-1}{2}, \quad C = 2p-2, \quad D = 2n.$$

Let $\Sigma := \left\{ (x_1,x_2) \in \mathbb{R}^2 \middle| x_1 \ge 0, x_2 > 0\right\}$ and consider the line

$$L := \left\{ (x_1,x_2) \in \Sigma \,\middle|\, -2n + \frac{p-1}{2}\frac{x_1}{x_2} = 0 \right\}.$$

Points in L are characterised by $x_2(t) = \frac{p-1}{4n}x_1(t) =: f$. This line splits Σ into two connected regions:

$$\begin{aligned} \Sigma_1 &:= \left\{ (x_1,x_2) \in \Sigma \,\middle|\, x_2 < f \right\}, \text{ and} \\ \Sigma_2 &:= \left\{ (x_1,x_2) \in \Sigma \,\middle|\, x_2 > f \right\}. \end{aligned}$$

In Σ_1, $x_2(t)$ is monotone increasing, while in Σ_2, $x_2(t)$ is monotone decreasing.

LEMMA 5.18. *The set Σ_2 is invariant under* (5.18).

PROOF. Let $(x_1(0), x_2(0)) \in \Sigma_2$, and assume that Σ_2 is not invariant. Since both $x_1(t)$ and $x_2(t)$ are monotone decreasing in Σ_2, there exists some $t_1 > 0$ such that $(x_1(t_1), x_2(t_1)) \in L$. However, on L, $\frac{\partial}{\partial t}x_2(t)\Big|_{t_1} = 0$ and

$$\frac{\partial}{\partial t}x_1(t)\Big|_{t_1} = 2-2p-\frac{16n^2(n+1-p)}{(p-1)^2} < 0.$$

Hence, $(x_1(t_1), x_2(t_1))$ instantly enters Σ_2. □

Given $(x_1(0),x_2(0)) \in \Sigma_2$, $x_1(t)$ and $x_2(t)$ are decreasing for all $t \geq 0$ by Lemma 5.18. We claim that $x_2(t)$ cannot be 0 before $x_1(t)$. Indeed, suppose there exists $T' < \infty$ such that $x_2(T') = 0$ yet $x_1(T') \neq 0$. This implies $(x_1(t),x_2(t)) \in \Sigma_1$, which is a contradiction since Σ_2 is invariant. Observe that

$$\frac{\partial}{\partial t}x_1(t) \leq 2-2p < 0.$$

Integrating this gives $x_1(t) \leq (2-2p)t + x_1(0)$, and so there exists $T < \frac{-x_1(0)}{2-2p} < \infty$ such that, as $t \to T$, $x_1(t) \to 0$ and $x_2(t)$ converges to a finite, non-negative limit.

On the other hand, if $(x_1(0),x_2(0)) \in \Sigma_1$, then $x_2(t)$ is increasing until some time $t_1 > 0$ where $(x_1(t_1),x_2(t_1)) \in L$. We have already seen that points in L instantly enter Σ_2. By the above argument, this implies there exists $T < \infty$ such that, as $t \to T$, $x_1(T) = 0$ and $x_2(t)$ converges to finite, non-negative limit.

Thus, given $x_1(0) \geq 0$ and $x_2(0) > 0$, there exists a unique solution to (5.18) defined on a maximal time interval $[0,T)$, and we see one of two possible behaviours:

(i) both $x_1(t)$ and $x_2(t)$ converge to 0 as $t \to T$, or

(ii) as $t \to T$, $x_1(t)$ converges to 0 and $x_2(t)$ converges to a finite, positive limit.

We first consider the situation in (i). Taylor's theorem implies that, around T, we may write

$$x_i(t) = x_i(T) + \frac{\partial}{\partial t}x_i(T) + \frac{\partial}{\partial t}x_i(T) + (T-t) + \frac{\frac{\partial^2}{\partial t^2}x_i(T)}{2!}(T-t)^2 + \cdots + \frac{\frac{\partial^k}{\partial t^{k_i}}x_i(T)}{k_i!}(T-t)^{k_i} + o((T-t)^{k_i})$$

for some $k_i \in \mathbb{Z}^+$, where $i = 1,2$. This implies that there exists some $n_i \in \mathbb{Z}^+$ and $c_i \in \mathbb{R}$ such that near T

$$x_i(t) = c_i(T-t)^{n_i} + o((T-t)^{n_i}). \tag{5.20}$$

By the boundedness of $\frac{x_1(t)}{x_2(t)}$, we conclude that $\frac{\partial}{\partial t}x_2(t)$ is bounded and $n_1 \geq n_2 > 0$. Since $\frac{\partial}{\partial t}x_2(t)$ is monotone and, as $t \to T$, $x_2(t)$ converges to 0, we find that $\frac{\partial}{\partial t}x_2(t)$ must converge to 0. With this in mind, by substituting (5.20) into (5.18) we find

$$\lim_{t\to T}\frac{\partial}{\partial t}x_2(t) = -2n + \frac{p-1}{2}\frac{c_1}{c_2}\lim_{t\to T}(T-t)^{n_1-n_2}.$$

Thus, $n_1 = n_2$. Since $(T-t)^{n_i} \geq 0$ for all t,

$$\int_0^t x_i(s)\mathrm{d}s = \int_0^t c_i(T-s)^{n_i}\mathrm{d}s + o\left(\int_0^t (T-s)^{n_i}\mathrm{d}s\right).$$

Moreover, we have that

$$\int_0^t \frac{\partial}{\partial s}x_2(s)\mathrm{d}s = \int_0^t -2n + \frac{p-1}{2}\frac{x_1(s)^2}{x_2(s)^2}\mathrm{d}s.$$

Hence, near T we may write

$$\begin{aligned} c_2(T-t)^{n_2} - x_2(0) &= \int_0^t -2n + \frac{p-1}{2}\frac{x_1(s)^2}{x_2(s)^2}\mathrm{d}s \\ &= -2nt - \frac{(p-1)c_1^2}{2c_2^2}\frac{(T-t)^{n_1-n_2+1}}{n_1-n_2+1} + \frac{(p-1)c_1^2T^{n_1-n_2+1}}{2c_2^2(n_1-n_2+1)} \\ &= -2nt - \frac{(p-1)c_1^2}{2c_2^2}(T-t) + \frac{(p-1)c_1^2}{2c_2^2}T. \end{aligned}$$

The right hand side is linear, implying $n_1 = n_2 = 1$. Therefore, $x_i(t)$ converges to 0 linearly in T for $i = 1,2$. Proposition 4.5 implies that T is a Type I singularity.

If we are in scenario (ii), then near T there exist constants $c_1, c_2 \in \mathbb{R}$, and $n \in \mathbb{Z}^+$ such that

$$x_1(t) = c_1(T-t)^{n_1} + o((T-t)^{n_1}),$$
$$x_2(t) = c_2 + o(1).$$

Again, as $(T-t)^{n_1}$ and 1 are positive, we know

$$\int_0^t x_1(s)\mathrm{d}s = \int_0^t c_1(T-s)^{n_1}\mathrm{d}s + o\left(\int_0^t (T-s)^{n_1}\mathrm{d}s\right), \text{ and}$$
$$\int_0^t x_2(s)\mathrm{d}s = \int_0^t c_2\mathrm{d}s + o\left(\int_0^t 1\mathrm{d}s\right).$$

Hence, near T

$$c_1(T-t)^{n_1} - x_1(0) + o((T-t)^{n_1}) = (2p-2)t - \frac{(n+1-p)c_1^2}{c_2^2(2n_1+1)}(T-t)^{2n_1+1} + \frac{(n+1-p)c_1^2}{c_2^2}\frac{T^{2n_1+1}}{2n_1+1}.$$

Equating the lower order terms implies $n_1 = 1$. Thus, $x_1(t)$ converges to 0 linearly in T. Proposition 4.5 implies that T is a Type I singularity. In addition, Theorem 4.8 implies that G/H converges to G/K in the Gromov-Hausdorff topology. This proves the following result.

THEOREM 5.19. *Let G/H be the homogeneous superspace* $\mathsf{SOSp}(2|2n)/\mathsf{U}(1|p-1)\times\mathsf{Sp}(2(n+1-p))$ *and consider a homogeneous G-invariant Riemannian metric of the form* (1.3) *on G/H. If $x_1(0) > 0$ and $x_2(0) > 0$, then there exists a unique solution to the homogeneous super Ricci flow defined on a maximal time interval $[0,T)$. Furthermore, $T < \infty$ is a Type I singularity, and we see one of two singular behaviours:*

(i) both $x_1(t)$ and $x_2(t)$ converge to 0 and the space G/H shrinks to a point;
(ii) $x_1(t)$ converges to 0 while $x_2(t)$ approaches a finite, positive limit. Moreover G/H converges to $\mathsf{SOSp}(2|2n)/\mathsf{SOSp}(2|2(p-1))\times\mathsf{Sp}(2(n+1-p))$ *in the Gromov-Hausdorff sense.*

Case II. Suppose $x_1(0) \geq 0$ and $x_2(0) < 0$. Then, by Lemma 5.16, $\frac{x_1(t)}{x_2(t)} \leq \frac{x_1(0)}{x_2(0)} \leq 0$ for all t such that a solution to (5.18) exists. We have the estimate

$$\frac{\partial}{\partial t}x_2(t) \leq -2n + \frac{p-1}{2}\frac{x_1(0)}{x_2(0)} =: C,$$

which integrated gives $x_2(t) \leq Ct + x_2(0)$. We claim that $x_2(t)$ cannot diverge to $-\infty$ in finite time. Indeed, suppose, for a contradiction, that there exists $T' < \infty$ such that $\lim_{t\to T'} x_2(t) = -\infty$. By the monotonicity of $x_1(t)$, for big enough t we have $-\frac{x_1(t)}{x_2(t)} \leq 1$, hence

$$\frac{\partial}{\partial t}x_2(t) \geq -2n + \frac{p-1}{2}.$$

Integrating the above, we find $x_2(t) \geq \left(-2n + \frac{p-1}{2}\right)t + x_2(0)$, which is a contradiction. Since $x_2(t)$ does not diverge to $-\infty$ in finite time, estimate (5.19) implies that there exists $T < \infty$ such that $x_1(T) = 0$ and x_2 converges to a negative, finite value. In this case, solutions enter quadrant three.

This proves the following result.

THEOREM 5.20. *Let G/H be the homogeneous superspace* $\mathsf{SOSp}(2|2n)/\mathsf{U}(1|p-1)\times\mathsf{Sp}(2(n+1-p))$ *and consider a homogeneous Riemannian metric of the form* (5.17) *on G/H. If $x_1(0) > 0$ and $x_2(0) < 0$, then there exists a unique solution to the homogeneous super Ricci flow defined on a maximal time interval $[0,T)$. The singular time T is characterised by $x_1(t) \to 0$ and $x_2(t)$ converging to a finite, negative value.*

Case III. Suppose $x_1(0) < 0$ and $x_2(0) > 0$. Then, by Lemma 5.16, $\frac{x_1(t)}{x_2(t)} < \frac{x_1(0)}{x_2(0)} < 0$ for all t such that a solution to (5.18) exists. We have that $x_2(t)$ is decreasing for all t since

$$\frac{\partial}{\partial t} x_2(t) < -2n + \frac{p-1}{2}\frac{x_1(0)}{x_2(0)} < 0.$$

The solution must stop after some finite time T, as otherwise $x_2(t)$ would become zero. We claim that $x_1(t)$ cannot diverge to $-\infty$ before $x_2(t)$ converges to 0. Suppose, for a contradiction, that there exists an initial condition and a time T' such that $\lim_{t\to T'} x_1(t) = -\infty$ and $x_2(T') = \alpha > 0$. Reparameterising by $\tau := -t$, we have that $x_1(\tau)$ and $x_2(\tau)$ are increasing. Let $x_1(0)$ be sufficiently negative, and $x_2(0) = \alpha$. Then, the solution in τ must intersect the original solution, which contradicts the uniqueness of solutions. Thus, there exists some $T < \infty$ such that $x_2(T) = 0$. As $t \to T$, $x_1(t)$ approaches a negative limit, possibly $-\infty$.

REMARK 5.21. *Considering the reparametrisation $\tau = -t$, we see that solutions with initial conditions $x_1(0) < 0$ and $x_2(0) > 0$ are increasing in τ. In fact,*

$$\frac{\partial}{\partial \tau} x_1(\tau) > 2p - 2 > 0,$$

and so $x_1(\tau) > (2p-2)\tau + x_1(0)$. Hence, all solutions hit $\{x_1 = 0\}$ in finite time. In other words, all solutions to (5.18) with initial conditions $x_1(0) < 0$ and $x_2(0) > 0$ come from solutions in quadrant one.

We have thus proven the following.

THEOREM 5.22. *Let G/H be the homogeneous superspace $\mathsf{SOSp}(2|2n)/\mathsf{U}(1|p-1)\times\mathsf{Sp}(2(n+1-p))$ and consider a homogeneous Riemannian metric of the form (5.17) on G/H. If $x_1(0) < 0$ and $x_2(0) > 0$, then there exists a unique solution to the homogeneous super Ricci flow defined on a maximal time interval $[0,T)$. The singular time T is characterised by $x_2(t) \to 0$ and $x_1(t)$ approaching a negative limit, possibly $-\infty$.*

Case IV. Define $\Sigma = \{(x_1,x_2) \in \mathbb{R}^2 : x_1, x_2 < 0\}$ and consider initial conditions $(x_1(0), x_2(0)) \in \Sigma$. As in Case I, we consider the line

$$L := \left\{ (x_1,x_2) \in \Sigma \,\middle|\, -2n + \frac{p-1}{2}\frac{x_1}{x_2} = 0 \right\}.$$

This again splits the domain into two connected regions:

$$\Sigma_1 := \left\{ (x_1,x_2) \in \Sigma \,\middle|\, x_2 < f \right\}, \text{ and}$$
$$\Sigma_2 := \left\{ (x_1,x_2) \in \Sigma \,\middle|\, x_2 > f \right\}.$$

However, Σ_1 is now characterised by $x_2(t)$ monotonically decreasing, while Σ_2 is characterised by $x_2(t)$ monotonically increasing. With the same proof as Lemma 5.18, we find Σ_2 to be invariant.

We consider three cases depending on the number of roots of (5.15):

(a) there are no roots of (5.15),

(b) there is exactly one positive root of (5.15), and

(c) there are two positive roots of (5.15).

Case (a). We first consider $(x_1(0), x_2(0)) \in \Sigma_1$. Here, both $x_1(t)$ and $x_2(t)$ are decreasing in t. We also see

$$\frac{\partial}{\partial t}x_1(t) > 2-2p-(n+1-p)\left(\frac{4n}{p-1}\right)^2, \text{ and}$$
$$\frac{\partial}{\partial t}x_2(t) > -2n.$$

In particular, neither $x_1(t)$ nor $x_2(t)$ can diverge to $-\infty$ in finite time. We claim that the solution must cross L. Indeed, if it doesn't, $\frac{x_1(t)}{x_2(t)}$ is bounded and increasing for all time, and so approaches a finite, positive limit. Denote this limit by $\bar{y}$. Consider the flow backwards in time by reparametrising $\tau = -t$. Assume there exists a solution with initial conditions $(x_1(0), x_2(0)) \in \Sigma_1$ such that $\frac{x_1(0)}{x_2(0)} = \bar{y}$. We know that $x_1(\tau)$ and $x_2(\tau)$ are monotone increasing in τ, so $(x_1(\tau), x_2(\tau))$ will cross the original solution. This contradicts uniqueness, and thus any solution with initial conditions in Σ_1 must cross L.

We now consider when $(x_1(0), x_2(0)) \in \Sigma_2$. Since Σ_2 is invariant, $(x_1(t), x_2(t)) \in \Sigma_2$ for all time. Moreover, $x_1(t)$ is always decreasing, while $x_2(t)$ is always increasing. Lemma 5.16 implies that $\frac{x_1(t)}{x_2(t)}$ is increasing in t due to (5.15) having no solutions. Hence,

$$\frac{\partial}{\partial t}x_2(t) > -2n + \frac{p-1}{2}\frac{x_1(0)}{x_2(0)} > 0,$$

which integrated gives $x_2(t) > \left(-2n + \frac{p-1}{2}\frac{x_1(0)}{x_2(0)}\right)t + x_2(0)$. The same proof as in Case III shows that $x_1(t)$ cannot diverge to $-\infty$ before $x_2(t)$ converges to 0. Consequently, there exists $T < \infty$ such that, as $t \to T$, $x_2(t)$ converges to 0 and $x_1(t)$ tends to a negative limit, which could be $-\infty$.

Case (b). Assume that $\bar{y}$ is the unique root of (5.15). As in the proof of Lemma 5.16, we may write

$$\frac{\partial}{\partial t}y(t) = -\frac{2n+1-p}{2x_2(t)}\left(y(t)-\bar{y}\right)^2.$$

We see that $y(t)$ is increasing for all t unless on the line $x_1(t) = \bar{y}x_2(t)$, which, since $\bar{y}$ is a positive root of (5.15), is contained in Σ_1 (see Figure 2). We remark that the line $x_1(t) = \bar{y}x_2(t)$ is invariant with both $x_1(t)$ and $x_2(t)$ decreasing in t. As a result, solutions along this line tend to the origin as $t \to -\infty$. We now consider two cases.

Suppose $y(0) < \bar{y}$. In particular, we assume a solution with initial conditions in Σ_1. By the uniqueness of solutions, $y(t) < \bar{y}$ for all t such that a solution exists. We know both $x_1(t)$ and $x_2(t)$ are decreasing,

$$\frac{\partial}{\partial t}x_1(t) > 2-2p-(n+1-p)\bar{y}^2, \text{ and}$$
$$\frac{\partial}{\partial t}x_2(t) > -2n.$$

Since the derivatives are bounded from below, the solution exists for all time. Moreover, $y(t)$ is increasing and bounded for all t, thus it approaches a positive limit, which must be $\bar{y}$.

Suppose now that $y(0) > \bar{y}$. We must consider separately when the initial conditions lie in Σ_1 and Σ_2. If we have a solution with initial conditions in Σ_2, the solution must stop in finite time as $x_2(t)$ converges to 0 and $x_1(t)$ approaches a negative limit, possibly $-\infty$. If we have a solution with initial conditions in Σ_1, we have shown in Case (a), that the solution crosses into Σ_2.

Case (c). Assume that $\overline{y_1}$ and $\overline{y_2}$ are distinct solutions to (5.15). Then, we may write

$$\frac{\partial}{\partial t}y(t) = -\frac{2n+1-p}{2x_2(t)}\left(y(t)-\overline{y_1}\right)\left(y(t)-\overline{y_2}\right).$$

The two lines $x_2(t) = \overline{y_1}x_1(t)$ and $x_2(t) = \overline{y_2}x_1(t)$ are invariant and contained in Σ_1. We now distinguish three different initial conditions (see Figure 3):

(i) $y(0) < \overline{y_1}$,

(ii) $\overline{y_1} < y(0) < \overline{y_2}$, and

(iii) $\overline{y_2} < y(0)$.

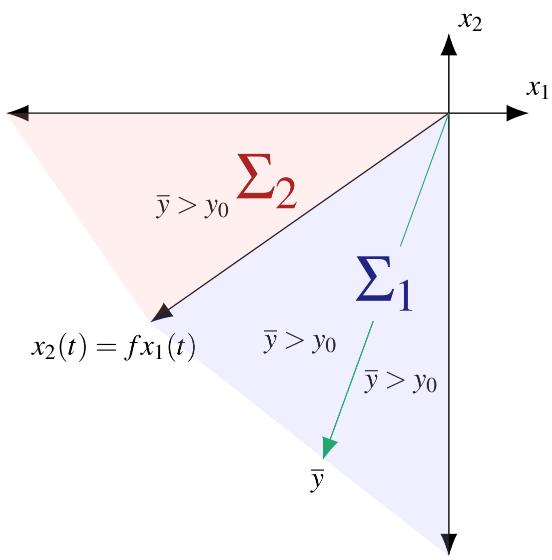

FIGURE 2. Quadrant 3 when there is exactly one positive root of (5.15)

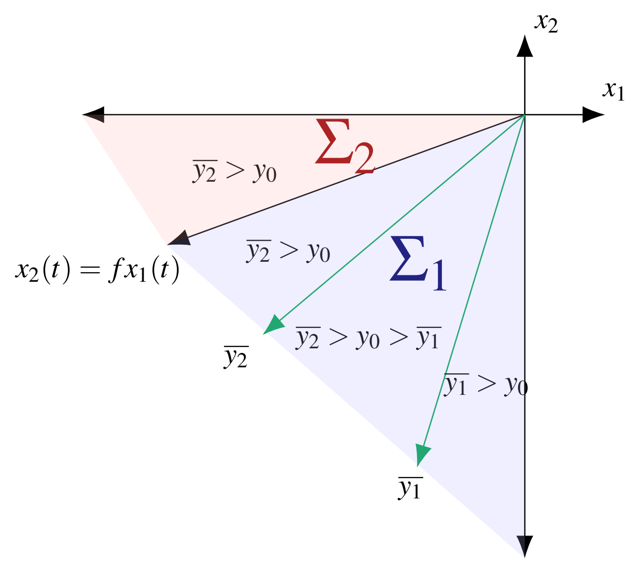

FIGURE 3. Quadrant 3 when there are two distinct positive roots of (5.15)

Case (i)*:* Here $y(0)$ is contained in Σ_1 and $y(t)$ is increasing in t. Indeed, we have

$$\frac{\partial}{\partial t}x_1(t) > 2 - 2p - (n+1-p)\overline{y_1}^2, \text{ and}$$
$$\frac{\partial}{\partial t}x_2(t) > -2n.$$

As such, there exists a solution for all time and we see $y(t) \to \overline{y_1}$ as $t \to \infty$.

Case (ii)*:* Again, $y(0) \in \Sigma_1$ and we have the bound $\overline{y_1} < y(t) < \overline{y_2}$ for all t such that a solution exists. Hence, $y(t)$ remains in Σ_1, where it is decreasing in t. We have the estimates

$$\frac{\partial}{\partial t}x_1(t) > 2 - 2p - (n+1-p)\overline{y_2}^2, \text{ and}$$
$$\frac{\partial}{\partial t}x_2(t) > -2n.$$

Hence, as $t \to \infty$, $y(t) \to \overline{y_1}$. Consider the reparametrisation $\tau = -t$. We find that $x_1(\tau), x_2(\tau)$ and $y(\tau)$ are increasing in τ. In fact,

$$\frac{\partial}{\partial \tau}x_2(\tau) > 2n - \frac{p-1}{2}\overline{y_1} > 0$$

implies that there exists $-\infty < T < 0$ such that $x_2(\tau) \to 0$ as $\tau \to T$. As $\overline{y_1} < y(\tau) < \overline{y_2}$ for all τ, we see $x_1(\tau) \to 0$ as $\tau \to T$ too.

Case (iii)*:* For $y(t)$ such that $(x_1(0), x_2(0)) \in \Sigma_1$, we have already shown that the solution must cross L. It suffices then to consider $(x_1(0), x_2(0)) \in \Sigma_2$. Since this is an invariant set, $y(t) \in \Sigma_2$ for all t such that a solution exists. As in Case (i), the solution stops in finite time when $x_2(t) \to 0$, and $x_1(t)$ tends to a negative limit (possibly $-\infty$). We have thus proven the following result.

THEOREM 5.23. *Let G/H be the homogeneous superspace $\mathsf{SOSp}(2|2n)/\mathsf{U}(1|p-1) \times \mathsf{Sp}(2(n+1-p))$ and consider a homogeneous Riemannian metric of the form* (5.17) *on G/H. If $x_1(0) < 0$ and $x_2(0) < 0$, then there exists a unique solution to the homogeneous super Ricci flow. We see different behaviour depending on the number of roots of* (5.15)*.*

- *(i) If* (5.15) *has no roots, then solutions exist on a maximal interval $[0, T)$ and, as $t \to T$, $x_2(t) \to 0$ while x_1 approaches a negative limit (possibly $-\infty$).*
- *(ii) If there is a unique root $\overline{y}$ of* (5.15)*, then we see one of two possible behaviours:*
  - *(a) if $\frac{x_1(0)}{x_2(0)} < \overline{y}$, then $\frac{x_1(t)}{x_2(t)}$ as $t \to \infty$;*
  - *(b) if $\frac{x_1(0)}{x_2(0)} > \overline{y}$, then there exists a finite extinction time T such that, as $t \to T$, $x_2(t) \to 0$ and $x_1(t)$ approaches a negative limit.*
- *(iii) If there are two roots, $\overline{y_1}$ and $\overline{y_2}$, to* (5.15)*, then we see one of three possible behaviours:*
  - *(a) if $\frac{x_1(0)}{x_2(0)} < \overline{y_1} < \overline{y_2}$, then $\frac{x_1(t)}{x_2(t)} \to \overline{y_1}$ as $t \to \infty$;*
  - *(b) if $\overline{y_1} < \frac{x_1(0)}{x_2(0)} < \overline{y_2}$, then there exists a finite time T such that $\frac{x_1(t)}{x_2(t)} \to \overline{y_1}$ as $t \to \infty$, and $\frac{x_1(t)}{x_2(t)} \to \overline{y_2}$ as $t \to -T$;*
  - *(c) if $\overline{y_1} < \overline{y_2} < \frac{x_1(0)}{x_2(0)}$, then there exists a finite time T such that, as $t \to T$, $x_2(t) \to 0$ and $x_1(t)$ approaches a negative limit that is possibly $-\infty$.*

5.7. Future directions

In this section, we discuss various questions and potential extensions stemming from the work presented in previous chapters. Perhaps the most natural problem to consider arises from our results in Theorem 5.19:

PROBLEM 1. *Renormalising the Ricci flow on homogeneous superspaces.*

Theorem 5.19 describes the existence of singularities in the Ricci flow for the homogeneous superspace $G/H = \mathsf{SOSp}(2|2n)/\mathsf{U}(1|p-1) \times \mathsf{Sp}(2(n+1-p))$. A natural next step is to consider renormalising the Ricci flow in an attempt to remove the singularities. Buzano [Buz14] remarks that in the classical case, renormalisation by volume only removes singularities when both $x_1(t)$ and $x_2(t)$ converge to 0. In these cases, the flow takes g to an Einstein metric. We suspect a similar conclusion in the super setting.

Following this, a natural goal is the resolution of Conjecture A. Moreover, we could consider when H is maximal in G. The overarching goal is to develop a general theory of the long time behaviour of the Ricci flow on homogeneous superspaces.

PROBLEM 2. *What is the long time behaviour of the Ricci flow of G-invariant metrics on compact homogeneous superspaces with more than two irreducible isotropy summands?*

Bibliography

[Arv03] A. Arvanitoyeorgos, *An introduction to Lie groups and the geometry of homogeneous spaces*, Student Mathematical Library, vol. 22, American Mathematical Society, Providence, RI, 2003. MR2011126

[AC09] A. Arvanitoyeorgos and I. Chrysikos, *Kähler C-spaces with two isotropy summands*, Tokyo J. Math **32** (2009), no. 2, 487–500.

[Bat80] M. Batchelor, *Two approaches to supermanifolds*, Trans. Amer. Math. Soc. **258** (1980), no. 1, 257–270, DOI 10.2307/1998294. MR0554332

[Bes87] A. L. Besse, *Einstein manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 10, Springer-Verlag, Berlin, 1987. MR0867684

[BWZ04] C. Böhm, M. Wang, and W. Ziller, *A variational approach for compact homogeneous Einstein manifolds*, Geom. Funct. Anal. **14** (2004), no. 4, 681–733, DOI 10.1007/s00039-004-0471-x. MR2084976

[Böh15] C. Böhm, *On the long time behavior of homogeneous Ricci flows*, Comment. Math. Helv. **90** (2015), no. 3, 543–571, DOI 10.4171/CMH/364. MR3420463

[Bou04] N. Bourbaki, *Lie groups and Lie algebras. Chapters 7–9*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 2004.

[BSV91] C. P. Boyer and O. A. Sánchez-Valenzuela, *Lie supergroup actions on supermanifolds*, Trans. Amer. Math. Soc. **323** (1991), no. 1, 151–175, DOI 10.2307/2001621. MR0998351

[Buz12] M. Buzano, *Topics in Ricci flow with symmetry*, PhD Thesis, Oxford University, 2012.

[Buz14] ———, *Ricci flow on homogeneous spaces with two isotropy summands*, Ann. Global Anal. Geom. **45** (2014), no. 1, 25–45, DOI 10.1007/s10455-013-9386-9. MR3152086

[CCF11] C. Carmeli, L. Caston, and R. Fioresi, *Mathematical foundations of supersymmetry*, EMS Series of Lectures in Mathematics, European Mathematical Society (EMS), Zürich, 2011. MR2840967

[CK04] B. Chow and D. Knopf, *The Ricci flow: an introduction*, Mathematical Surveys and Monographs, vol. 110, American Mathematical Society, Providence, RI, 2004. MR2061425

[Col67] S. Coleman, *All Possible Symmetries of the S Matrix*, Phys. Rev. **159** (1967), 1251–1256.

[DM99] P. Deligne and J. W. Morgan, *Notes on supersymmetry (following Joseph Bernstein)*, Quantum fields and strings: a course for mathematicians, Vol. 1, 2 (Princeton, NJ, 1996/1997), Amer. Math. Soc., Providence, RI, 1999, pp. 41–97. MR1701597

[DeT83] D. M. DeTurck, *Deforming metrics in the direction of their Ricci tensors*, J. Differential Geom. **18** (1983), no. 1, 157–162. MR0697987

[DeW84] B. DeWitt, *Supermanifolds*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, 1984. MR0778559

[DK08] W. Dickinson and M. M. Kerr, *The geometry of compact homogeneous spaces with two isotropy summands*, Ann. Global Anal. Geom. **34** (2008), no. 4, 329–350, DOI 10.1007/s10455-008-9109-9. MR2447903

[EH00] D. Eisenbud and J. Harris, *The geometry of schemes*, Graduate Texts in Mathematics, vol. 197, Springer-Verlag, New York, 2000. MR1730819

[Goe08] O. Goertsches, *Riemannian supergeometry*, Math. Z. **260** (2008), no. 3, 557–593, DOI 10.1007/s00209-007-0288-z. MR2434470

[GPRZ23] M. Gould, A. Pulemotov, J. Rasmussen, and Y. Zhang, *The Ricci curvature and Einstein metrics on homogeneous superspaces* (2023), in preparation.

[Gri13] David J Griffiths, *Introduction to electrodynamics*, Pearson, 2013.

[HŁS75] R. Haag, J. Łopuszański, and M. Sohnius, *All possible generators of supersymmetries of the S-matrix*, Nuclear Physics B **88** (1975), no. 2, 257-274.

[Ham84] R. S. Hamilton, *The Ricci curvature equation*, Seminar on nonlinear partial differential equations (Berkeley, Calif., 1983), Math. Sci. Res. Inst. Publ., vol. 2, Springer, New York, 1984, pp. 47–72, DOI 10.1007/978-1-4612-1110-5-4. MR0765228

[Ham82] ______, *Three-manifolds with positive Ricci curvature*, J. Differential Geometry **17** (1982), no. 2, 255–306. MR0664497
[Har77] R. Hartshorne, *Algebraic geometry.*, Springer-Verlag, New York-Heidelberg,, 1977. MR0463157
[IJ92] J. Isenberg and M. Jackson, *Ricci flow of locally homogeneous geometries on closed manifolds*, J. Differential Geom. **35** (1992), no. 3, 723–741. MR1163457
[Kac77] V. G. Kac, *Lie superalgebras*, Advances in Math. **26** (1977), no. 1, 8–96, DOI 10.1016/0001-8708(77)90017-2. MR0486011
[Kim90] M. Kimura, *Homogeneous Einstein metrics on certain Kähler C-spaces*, Adv. Stud. Pure. Math. **18** (1990), no. 1, 303-320.
[KN63] S. Kobayashi and K. Nomizu, *Foundations of differential geometry. Vol I.*, Interscience Publishers (a division of John Wiley & Sons, Inc.), New York-London, 1963. MR0152974
[KN69] ______, *Foundations of differential geometry. Vol. II.*, Interscience Publishers John Wiley & Sons, Inc., New York-London-Sydney, 1969. MR0238225
[Kos77] B. Kostant, *Graded manifolds, graded Lie theory, and prequantization.*, Differential geometrical methods in mathematical physics (Proc. Sympos., Univ. Bonn, Bonn, 1975), Lecture Notes in Math., Vol. 570, , 1977, pp. 177–306. MR0580292
[Laf15] R. A. Lafuente, *Scalar curvature behavior of homogeneous Ricci flows*, J. Geom. Anal. **25** (2015), no. 4, 2313–2322, DOI 10.1007/s12220-014-9514-1. MR3427126
[Lee18] J. M. Lee, *Introduction to Riemannian manifolds*, 2nd ed., Graduate Texts in Mathematics, vol. 176, Springer, Cham, 2018. MR3887684
[Lei80] D. A. Leites, *Introduction to the theory of supermanifolds.*, Uspekhi Math. Nauk **no. 1(211),** (1980), 3–57, 255 (Russian). MR0565567
[Noe18] E. Noether, *Invariante Variationsprobleme*, Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse (1918), 235-257 (ger).
[Pet06] P. Petersen, *Riemannian geometry*, 2nd ed., Graduate Texts in Mathematics, vol. 171, Springer, New York, 2006. MR2243772
[PR19] A. Pulemotov and Y. Rubinstein, *Ricci iteration on homogeneous spaces*, Trans. Amer. Math. Soc. **371** (2019), no. 9, 6257–6287.
[Rog07] A. Rogers, *Supermanifolds*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2007. MR2320438
[Rog80] ______, *A global theory of supermanifolds*, J. Math. Phys. **21** (1980), no. 6, 1352–1365, DOI 10.1063/1.524585. MR0574696
[Sch84] T. Schmitt, *Super differential geometry*, Report MATH, vol. 84-5, Akademie der Wissenschaften der DDR, Institut für Mathematik, Berlin, 1984. MR0786297
[Var84] V.S. Varadarajan, *Lie groups, Lie algebras, and their representations*, Graduate Texts in Mathematics, vol. 102, Springer New York, NY, 1984.
[Var04] V. S. Varadarajan, *Supersymmetry for mathematicians: an introduction*, Courant Lecture Notes in Mathematics, vol. 11, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2004. MR2069561
[WZ86] M. Y. Wang and W. Ziller, *Existence and nonexistence of homogeneous Einstein metrics*, Invent. Math. **84** (1986), no. 1, 177–194, DOI 10.1007/BF01388738. MR0830044
[WZ91] M. Wang and W. Ziller, *On isotropy irreducible Riemannian manifolds*, Acta Math. **166** (1991), no. 3-4, 223–261, DOI 10.1007/BF02398887. MR1097024
[Wol68] J. Wolf, *The geometry and structure of isotropy irreducible homogeneous spaces*, Acta. Math. **120** (1968), 59—148.

APPENDIX A

Mathematica code for the computation in Proposition 5.7:

```
s = 2; "Number of summands"
f[OrderlessPatternSequence[a_, b_, c_]] = g[a, b, c]; "Symmetry of [ijk]"
d = Table[Symbol["d" <> ToString@i], {i, s}]; "Creates an array of variables di"
x = Table[Symbol["x" <> ToString@i], {i, s}]; "Creates an array of variables xi"
r = Table[
b/(2 Part[x, i]) +
1/(2 Part[d, i]) Sum[
Part[x, k]/(Part[x, j] Part[x, i]) f[i, j, k], {j, 1, s}, {k, 1,
  s}] -
1/(4 Part[d, i]) Sum[
Part[x, i]/(Part[x, k] Part[x, j]) f[i, j, k], {j, 1, s}, {k, 1,
  s}], {i, 1, s}]; "Creates an array of the Ricci coefficients"
S = Sum[Part[r, i]*Part[d, i], {i, 1, s}]; "Scalar curvature expression"
dx = Table[-2 Part[r, i]*Part[x, i] +
2/(Sum[Part[d, i], {i, 1, s}])*S*Part[x, i], {i, 1, s}]; "Normalised Ricci flow"
Formula =
2 (Sum[Part[r, i]^2*Part[d, i], {i, 1, s}] -
1/(Sum[Part[d, i], {i, 1, s}]) S^2); "Proposed formula for the evolution of S"
dS = Sum[-1/2 b Part[d, m] Part[dx, m]/Part[x, m]^2 +
Part[dx, m]/2 (
Sum[(Sum[(-Part[x, k]/(Part[x, m]^2 Part[x, i]) f[i, m, k])*
Boole[k != m] , {k, 1, s}] +
Sum[(1/(Part[x, i] Part[x, j]) f[i, j, m])*
Boole[j != m], {j, 1, s}])*Boole[i != m], {i, 1, s}] +
Sum[(Sum[(-Part[x, k]/(Part[x, j] Part[x, m]^2) f[m, j, k])*
Boole[k != m], {k, 1, s}])*Boole[j != m], {j, 1, s}] +
Sum[(-2 Part[x, k]/Part[x, m]^3 f[m, m, k])*Boole[k != m], {k,
  1, s}] - 1/Part[x, m]^2 f[m, m, m]) -
Part[dx, m]/
4 (Sum[(Sum[(-Part[x, i]/(Part[x, k] Part[x, m]^2) f[i, m, k])*
Boole[k != m], {k, 1, s}] +
Sum[(-Part[x, i]/(Part[x, j] Part[x, m]^2) f[i, j, m])*
Boole[j != m], {j, 1, s}] -
2 Part[x, i]/Part[x, m]^3 f[i, m, m])*Boole[i != m], {i, 1,
  s}] +
Sum[(Sum[(1/(Part[x, j] Part[x, k]) f[m, j, k])*
Boole[k != m], {k, 1, s}])*Boole[j != m], {j, 1, s}] -
1/Part[x, m]^2 f[m, m, m]), {m, 1, s}]; "Time derivative of S"
FullSimplify[Formula - dS]
```